COMPUTER SCIENCE TRIPOS Part IA – 2019 – Paper 1

5 Numerical Analysis (abr28)

(*a*) Let f be a single variable real function that has at least one root α, and that admits a Taylor expansion everywhere.

(*i*) Starting from the truncated form of the Taylor expansion of $f(x)$ about x_n, derive the recursive expression for the Newton-Raphson (NR) estimate x_n of the root at the $(n+1)$th step. [1 mark]

(*ii*) Consider the general Taylor expansion of $f(\alpha)$ about x_n. Using big O notation for an appropriate Taylor remainder and denoting the NR error at the nth step by e_n, prove that the NR method has quadratic convergence rate. That is, show that e_{n+1} is proportional to e_n^2 plus a bounded remainder. State the required conditions for this to hold, paying attention to the interval spanned during convergence. [6 marks]

(*iii*) Briefly explain two of the known problems of the NR method from an implementation standpoint or otherwise. [2 marks]

(*b*) Let $f(x) = x^2 - 1$. Suppose we wish to find the positive root of f using the Newton-Raphson (NR) method starting from an initial guess $x_0 \geq 1$.

(*i*) Show that if $x_0 \geq 1$ then $x_n \geq 1$ for all $n \geq 1$. [3 marks]

(*ii*) Thus find an upper bound for NR's x_{n+1} estimate in terms of x_n and in turn find an upper bound for x_n in terms of x_0. [5 marks]

(*iii*) Using the above, estimate the number of NR iterations to obtain the root with accuracy 10^{-9} for a wild initial guess $x_0 = 10^9$. [*Hint:* You may wish to approximate 10^3 by 2^{10}.] [3 marks]