## 8 Semantics of Programming Languages (PMS)

Consider the following syntax up to alpha equivalence, where $n$ ranges over natural numbers, $x$ over a set of variables, and (as usual) $x$ is binding in $e$ in $\mathbf{fn}\, x \Rightarrow e$.

$$\begin{array}{ll} \textit{expressions},\ e ::= & n \mid x \mid \mathbf{fn}\, x \Rightarrow e \mid e\, e' \\ \textit{values},\ v ::= & n \mid x \mid \mathbf{fn}\, x \Rightarrow e \end{array}$$

(*a*) Define free variables $\mathrm{fv}(e)$ and capture-avoiding substitution $\{e/z\}e'$. [3 marks]

(*b*) Define a left-to-right call-by-value reduction relation $e \longrightarrow e'$. [3 marks]

Implementing a language using substitution is inefficient, as each substitution has to traverse a potentially large subterm. Consider the following proposal for an abstract machine for this language using environments $E$, lists of variable/value pairs.

$$\boxed{\langle E, e \rangle \longrightarrow \langle E', e' \rangle}$$

$$\frac{(x, v) \in E}{\langle E, x \rangle \longrightarrow \langle E, v \rangle} \ \text{LOOKUP}$$

$$\frac{x \notin \mathrm{dom}(E) \cup \mathrm{fv}(\mathrm{range}(E)) \cup \mathrm{fv}(v)}{\langle E, (\mathbf{fn}\, x \Rightarrow e)\, v \rangle \longrightarrow \langle (x, v) :: E, e \rangle} \ \text{FN}$$

$$\frac{\langle E, e_1 \rangle \longrightarrow \langle E', e_1' \rangle}{\langle E, e_1\, e_2 \rangle \longrightarrow \langle E', e_1'\, e_2 \rangle} \ \text{APP\_LEFT}$$

$$\frac{\langle E, e_2 \rangle \longrightarrow \langle E', e_2' \rangle}{\langle E, v_1\, e_2 \rangle \longrightarrow \langle E', v_1\, e_2' \rangle} \ \text{APP\_RIGHT}$$

(*c*) Give the sequence of abstract-machine reduction steps, including the configurations and the names of the rules used, for the initial configuration below. You need not give full derivation trees.

$$\langle [], ((\mathbf{fn}\, x \Rightarrow (\mathbf{fn}\, y \Rightarrow x\, y))\, (\mathbf{fn}\, z \Rightarrow z))\, 3 \rangle$$

[5 marks]

(*d*) Explain, with a concrete example and its reduction sequence, what could go wrong if the premise of FN had been omitted. [5 marks]

(*e*) Write $\{E\}e$ for the iterated substitution defined by

$$\begin{array}{lcl} \{[]\}e & = & e \\ \{(x, v) :: E\}e & = & \{E\}(\{v/x\}e) \end{array}$$

Prove that $\{E\}(e_1\, e_2) = (\{E\}e_1\, \{E\}e_2)$. [4 marks]