7 Denotational Semantics (MPF)

(*a*) (*i*) Define the notion of least pre-fixed point $\mathsf{fix}(f)$ of a continuous endofunction f on a domain and state Tarski's fixed point theorem for it. [2 marks]

(*ii*) State Scott's fixed point induction principle. [2 marks]

(*b*) Let $f : D \to D$, $g : E \to E$, and $h : D \to E$ be continuous functions between domains such that $h \circ f = g \circ h : D \to E$. Show that if h is strict (that is, $h(\bot_D) = \bot_E$) then $\mathsf{fix}(g) = h\big(\mathsf{fix}(f)\big)$. [4 marks]

(*c*) Let Σ^* denote the set of strings over an alphabet Σ, and let $\mathcal{P}(\Sigma^*)$ be the domain of all subsets of Σ^* ordered by inclusion.

Consider the continuous functions

$$\begin{array}{lcll} \text{(union)} & + & : & \mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*) \\ \text{(concatenation)} & \cdot & : & \mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*) \end{array}$$

given, for all $X, Y \subseteq \Sigma^*$, by

$$\begin{array}{rcl} X + Y & = & \{\, w \in \Sigma^* \mid w \in X \text{ or } w \in Y \,\} \\ X \cdot Y & = & \{\, uv \in \Sigma^* \mid u \in X \text{ and } v \in Y \,\} \end{array}$$

(*i*) Using part (*b*), or otherwise, show that, for all $A, B, C \subseteq \Sigma^*$,

(1) $\mathsf{fix}\big(\lambda X.\, C \cdot B + A \cdot X\big) \;=\; \mathsf{fix}\big(\lambda X.\, C + A \cdot X\big) \cdot B$ [4 marks]

(2) $\mathsf{fix}\big(\lambda X.\, A \cdot C + A \cdot B \cdot X\big) \;=\; A \cdot \mathsf{fix}\big(\lambda X.\, C + B \cdot A \cdot X\big)$ [4 marks]

(*ii*) For $P \subseteq \Sigma^*$, let $P^\star \subseteq \Sigma^*$ be defined as

$$P^\star \;=\; \mathsf{fix}\big(\lambda X.\, E + P \cdot X\big)$$

where E denotes the singleton set $\{\varepsilon\}$ consisting of the empty string ε.

Using part (*c*)(*i*), or otherwise, show that, for all $S, T \subseteq \Sigma^*$,

$$(S \cdot T)^\star \cdot S \;=\; S \cdot (T \cdot S)^\star$$ [4 marks]