5 Denotational Semantics (MPF)

For all PCF types τ and all $M \in \mathrm{PCF}_{\tau \to bool}$, let $M^{\#} \subseteq [\![\tau]\!]$ be defined as

$$M^{\#} = \{\, d \in [\![\tau]\!] \mid [\![M]\!](d) = true \,\}$$

Indicate whether the following statements are true or false, respectively providing a proof or a counterexample. You may use any standard results provided that you state them clearly.

(*a*) For all PCF types τ and all $M, N \in \mathrm{PCF}_{\tau \to bool}$, if $M^{\#} \subseteq N^{\#}$ then $\vdash M \leq_{\mathrm{ctx}} N : \tau$. [5 marks]

(*b*) For all PCF types τ and all $M, N \in \mathrm{PCF}_{\tau \to bool}$, if $\vdash M \leq_{\mathrm{ctx}} N : \tau$ then $M^{\#} \subseteq N^{\#}$. [5 marks]

(*c*) For all PCF types τ and all $M, N \in \mathrm{PCF}_{\tau \to bool}$, there exists $P \in \mathrm{PCF}_{(\tau \to bool) \to ((\tau \to bool) \to (\tau \to bool))}$ such that $(P\,M\,N)^{\#} = M^{\#} \cap N^{\#}$. [5 marks]

(*d*) For all PCF types τ and all $M, N \in \mathrm{PCF}_{\tau \to bool}$, there exists $P \in \mathrm{PCF}_{(\tau \to bool) \to ((\tau \to bool) \to (\tau \to bool))}$ such that $(P\,M\,N)^{\#} = M^{\#} \cup N^{\#}$.

Hint: Consider the fact, for which you need not provide a proof, that there is no PCF-definable function $f \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot))$ such that $f\,true\,\bot = f\,\bot\,true = true$ and $f\,false\,false \neq true$. [5 marks]