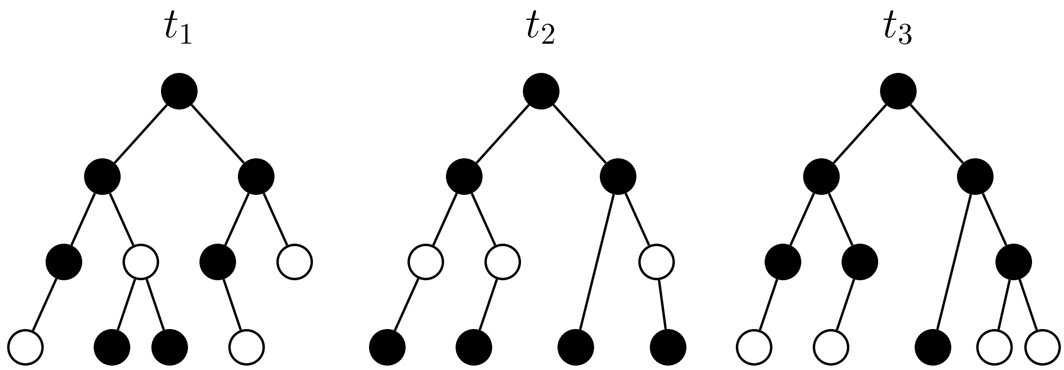
## 7 Algorithms (FMS)

*Reminders:* A red-black tree has leaf nodes (black) and may have non-leaf nodes (red or black). The height of a red-black tree is the number of edges on the longest path from the root to any leaf.
*Mathematical hint:* $\sum_{j=1}^{k} 2^{-j} = 1 - 2^{-k}$

(*a*) Indicate whether each of the following trees is or is not a valid red-black tree. Justify your answers with reference to the defining invariants of red-black trees. You may, but do not have to, redraw the trees if it helps you clarify a point. [8 marks]




(*b*) Let $r(h), b(h), l(h)$ respectively represent the number of red non-leaf nodes, the number of black non-leaf nodes, and the number of leaf nodes in a red-black tree as a function of the height $h$ of the tree. Under each of the conditions stated below, and assuming that the tree has as few red nodes as possible, derive mathematical expressions for $r(h), b(h), l(h)$, preferably in closed form. Clearly justify your answers, with drawings if appropriate. Expressions that are valid only for even (or odd) values of $h$ can still earn full marks if properly derived and explained.

(*i*) Derive the $r(h), b(h), l(h)$ expressions assuming that the red-black tree has the largest possible number of nodes for a given height $h$. [6 marks]

(*ii*) Derive the $r(h), b(h), l(h)$ expressions assuming that the red-black tree has the smallest possible number of nodes for a given height $h$. [6 marks]