6 Denotational Semantics (MPF)

For partially ordered sets $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$, define the set

$$(P \Rightarrow Q) = \{f \mid f \text{ is a monotone function from } (P, \sqsubseteq_P) \text{ to } (Q, \sqsubseteq_Q)\}$$

and, for all $f, g \in (P \Rightarrow Q)$, let

$$f \sqsubseteq_{(P \Rightarrow Q)} g \iff \forall\, p \in P.\, f(p) \sqsubseteq_Q g(p)$$

(*a*) Let $(P, \sqsubseteq_P)$ and $(Q, \sqsubseteq_Q)$ be partially ordered sets.

(*i*) Prove that $\big((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)}\big)$ is a partially ordered set. [4 marks]

(*ii*) Prove that if $(Q, \sqsubseteq_Q)$ is a domain then so is $\big((P \Rightarrow Q), \sqsubseteq_{(P \Rightarrow Q)}\big)$. [6 marks]

(*b*) For $\mathbb{N}$ the set of natural numbers partially ordered by the equality relation and for $S_\perp$ the flat domain determined by a set S, consider the domain $\big((\mathbb{N} \Rightarrow S_\perp), \sqsubseteq_{(\mathbb{N} \Rightarrow S_\perp)}\big)$.

(*i*) A function $f \in (\mathbb{N} \Rightarrow S_\perp)$ is said to be *finite* whenever the subset of $\mathbb{N}$ given by $\{n \mid f(n) \neq \perp\}$ is finite.

Show that every function in $(\mathbb{N} \Rightarrow S_\perp)$ is the least upper bound of a countable chain of finite functions. [4 marks]

(*ii*) For a domain $(D, \sqsubseteq)$, an element $d \in D$ is said to be *isolated* (with respect to $\sqsubseteq$) whenever, for all countable chains $(x_0 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots)$ in D with $d \sqsubseteq \bigsqcup_{n \geq 0} x_n$, there exists $m \geq 0$ with $d \sqsubseteq x_m$.

Prove that a function in $(\mathbb{N} \Rightarrow S_\perp)$ is isolated (with respect to $\sqsubseteq_{(\mathbb{N} \Rightarrow S_\perp)}$) iff it is finite. [6 marks]