8 Mathematical Methods for Computer Science (RJG)

Suppose that X is a random variable with moment generating function $M_X(t)$ which you may assume is well-defined and finite for all t.

(*a*) Show that for any constant a and for all $t \geq 0$

$$\mathbb{P}(X \geq a) \leq e^{-ta} M_X(t)\,.$$

[5 marks]

(*b*) Show that for any constant a

$$\mathbb{P}(X \geq a) \leq e^{-f(a)}$$

where

$$f(a) = \max_{t \geq 0} \left(ta - \ln M_X(t)\right)\,.$$

[5 marks]

(*c*) Let $X_1, X_2, \ldots$ be a sequence of independent random variables each with the same distribution as X. Show that for any $a > \mathbb{E}(X)$

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq a\right) \leq e^{-nf(a)}\,.$$

[5 marks]

(*d*) Show that $\mathbb{P}(X \geq a) \leq e^{-a^2/2}$ when $X \sim N(0,1)$ is a standard Normal random variable and $a > 0$. You may use the result that in this case $M_X(t) = e^{t^2/2}$.

[5 marks]