10 Semantics of Programming Languages (SS)

This question is about a variation on a fragment of the L2 language in which functions take two arguments. The language has the following expressions:

$$e \quad ::= \quad x \;\mid\; \mathsf{fn}\,(x_1, x_2) \Rightarrow e \;\mid\; e_0\,(e_1, e_2) \;\mid\; n$$

where x ranges over variables and n ranges over integers. As usual, $\mathsf{fn}\,(x, y) \Rightarrow e$ is binding: we work up-to α-equivalence and require that x and y are different.

(*a*) Write down a call-by-name operational semantics for this language. [2 marks]

(*b*) Consider the following type system. The types are

$$T \;::=\; \mathsf{int} \mid \mathsf{ret} \mid (T_1, T_2) \rightarrow \mathsf{ret}$$

A context Γ is a finite partial function from variables to types. The type system is given by the following rules:

$$\frac{-}{\Gamma, x : T, \Gamma' \vdash x : T} \qquad \frac{\Gamma \vdash e_0 : (T_1, T_2) \rightarrow \mathsf{ret} \quad \Gamma \vdash e_1 : T_1 \quad \Gamma \vdash e_2 : T_2}{\Gamma \vdash e_0\,(e_1, e_2) : \mathsf{ret}}$$

$$\frac{-}{\Gamma \vdash n : \mathsf{int}}\ (n \text{ is an integer}) \qquad \frac{\Gamma, x_1 : T_1, x_2 : T_2 \vdash e : \mathsf{ret}}{\Gamma \vdash \mathsf{fn}\,(x_1, x_2) \Rightarrow e : (T_1, T_2) \rightarrow \mathsf{ret}}$$

(The idea is that $(T_1, T_2) \rightarrow \mathsf{ret}$ is a type of functions taking arguments of type T_1 and T_2. However, there are no expressions of type ret in the empty context, and so rather than returning a result you pass it to a 'continuation'.)

(*i*) Find a type T for which $\vdash \mathsf{fn}\,(x, k) \Rightarrow k\,(3, x) : T$, giving a derivation. [3 marks]

(*ii*) Give a derivation of the following judgement: [2 marks]

$$k : (\mathsf{int}, \mathsf{ret}) \rightarrow \mathsf{ret} \vdash \mathsf{fn}\,(x, l) \Rightarrow l\,(7, k) : (\mathsf{int}, (\mathsf{int}, (\mathsf{int}, \mathsf{ret}) \rightarrow \mathsf{ret}) \rightarrow \mathsf{ret}) \rightarrow \mathsf{ret}$$

(*c*) Prove the following 'progress' theorem for this language: [6 marks]

If $\vdash e : T$ then either $e = (\mathsf{fn}\,(x, y) \Rightarrow e')$, or e is an integer, or there is e' such that $e \longrightarrow e'$.

(*d*) We now consider the situation where there is a type posint of positive integers which is a subtype of int.

Define a subtyping relation $<:$ and extend the type system to accommodate it. Demonstrate it by giving a derivation of the following judgement:

$$k : (\mathsf{int}, \mathsf{ret}) \rightarrow \mathsf{ret} \vdash \mathsf{fn}\,(x, l) \Rightarrow l\,(7, k) : (\mathsf{int}, (\mathsf{int}, (\mathsf{posint}, \mathsf{ret}) \rightarrow \mathsf{ret}) \rightarrow \mathsf{ret}) \rightarrow \mathsf{ret}$$

[7 marks]