5   Discrete Mathematics II (GW)

Let A, B, C be sets with relations $R \subseteq A \times B$ and $S \subseteq B \times C$.

(a) Give the definition of the relational composition $S \circ R$.   [1 mark]

(b) The functions p, q, r are defined as follows:

$$p : A \times B \times C \to A \times B \quad \text{such that} \quad p(a, b, c) = (a, b)$$
$$q : A \times B \times C \to B \times C \quad \text{such that} \quad q(a, b, c) = (b, c)$$
$$r : A \times B \times C \to A \times C \quad \text{such that} \quad r(a, b, c) = (a, c)$$

(i)   Describe the inverse images $p^{-1}R$ and $q^{-1}S$.   [2 marks]

(ii) For $X \subseteq A \times B \times C$ describe its direct image $r\,X$ under r.   [2 marks]

(iii) Prove that the relational composition $S \circ R$ equals $r\,(p^{-1}R \cap q^{-1}S)$, the direct image of the set $p^{-1}R \cap q^{-1}S$ under r.   [5 marks]

(c) Suppose the relations R and S are countable. Is the relational composition $S \circ R$ countable? Justify your answer. [*Note:* You may use any well-known results provided you state them clearly.]   [6 marks]

(d) Suppose now that $A = B = C$ and that both R and S are well-founded relations. Is the relational composition $S \circ R$ well-founded? Justify your answer.

[4 marks]