2011 Paper 8 Question 14

Types

Terms in the polymorphic lambda calculus (PLC) are given by the grammar

$$M ::= x \quad | \quad \lambda x : \tau(M) \quad | \quad M\ M \quad | \quad \Lambda\alpha(M) \quad | \quad M\ \tau$$

where τ is a type, α a type variable and x a variable.

(*a*) Give the rules for the type system in PLC. [2 marks]

(*b*) Give the rules for the relation, $\rightarrow_\beta$, of beta-reduction in PLC. Explain what it means for a term to be beta-normal. [3 marks]

(*c*) Terms in *head-normal form*, H, can be described in PLC by the grammar

$$\begin{aligned} A &::= x \quad | \quad A\ H \quad | \quad A\ \tau \\ H &::= A \quad | \quad \lambda x : \tau(H) \quad | \quad \Lambda\alpha(H) \end{aligned}$$

Arguing by induction on the structure of terms, or otherwise, prove that every term in PLC that is both typable and beta-normal is of head-normal form. [8 marks]

(*d*) Natural numbers can be encoded into PLC using the type *nat* defined as

$$nat \stackrel{\text{def}}{=} \forall\alpha(\alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha)$$

The encoding uses the Church-numerals defined as

$$\overline{n} \stackrel{\text{def}}{=} \Lambda\alpha(\lambda x{:}\alpha(\lambda y{:}\alpha \rightarrow \alpha(\underbrace{y(y(y \ldots (y}_{n \text{ occurrences}}\ x) \ldots)))))$$

Prove that the Church-numerals are the only closed, beta-normal terms of type *nat*.

Hint: Use the result proved in part (*c*) and do a case analysis over the form of terms in head-normal form. You may assume without proof that if $\Gamma \vdash M : \tau$ is provable in the PLC type system, then the free variables of the term M are contained in the domain of the typing environment Γ.

[7 marks]