2010 Paper 9 Question 8

Information Theory and Coding

(*a*) Show that for an alphabet of unlimited size, it is possible to create a uniquely decodable prefix code for its letters such that their average codeword length is no more than 2 bits, provided that the alphabet letters obey a certain probability distribution. (No letter has probability 0; the letters' probabilities all sum up to 1; and "average codeword length" weights each codeword length by the probability of occurrence of the letter.) Give the necessary descending sequence of letter probabilities, and a uniquely decodable codeword with the prefix property for each letter, and show that the average codeword length remains ≤ 2 bits regardless of how large the alphabet may be. [6 marks]

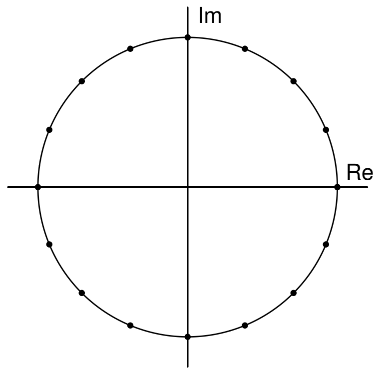
Hint: you may invoke without proof the following series limit:

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} \log_2(2^n) = 2$$

(*b*) Why is the JPEG2000 image compression protocol so much better than the original JPEG in delivering good results even at severe compression rates? What are the main differences in the basis functions or coding wavelets used, what reconstruction artefacts are thereby avoided, and why? [4 marks]

(*c*) The signal-to-noise ratio SNR of a continuous communication channel can be different in different parts of its frequency range. For example, the noise might be predominantly high frequency hiss, or low frequency rumble. Explain how the information capacity C of a noisy continuous communication channel, whose available band is the frequency range from ω_1 to ω_2, may be defined in terms of its signal-to-noise ratio as a function of frequency, SNR(ω). Define the bit rate for this noisy channel's information capacity, C, in bits/second, in terms of this SNR(ω) function of frequency. [6 marks]

(*d*) Suppose a discrete data sequence $\{g_n\}$ consisting of 16 data points is Fourier transformed using an FFT algorithm. The fourth frequency component is a complex exponential that evolves four cycles over the course of the 16-point data sequence. Using a unit circle diagram in the complex plane as shown here, explain why the 16 complex multiplications implicitly needed to compute any Fourier coefficient of this data can be reduced to just 4 multiplications for this coefficient, by first adding together certain data points in $\{g_n\}$.



[4 marks]