2010 Paper 9 Question 15

Types

Consider the following type and expressions of the Polymorphic Lambda Calculus (PLC):

$$\begin{aligned} nat &= \forall\alpha(\alpha \to (\alpha \to \alpha) \to \alpha) \\ Z &= \Lambda\alpha(\lambda x : \alpha(\lambda f : \alpha \to \alpha\,(x))) \\ S &= \lambda y : nat(\Lambda\alpha(\lambda x : \alpha(\lambda f : \alpha \to \alpha\,(f(y\,\alpha\,x\,f))))) \end{aligned}$$

(*a*) What are the types of Z and S? [2 marks]

(*b*) Show that there is a closed PLC expression I of type

$$\forall\alpha(\alpha \to (\alpha \to \alpha) \to nat \to \alpha)$$

satisfying the following beta-conversions:

$$\begin{aligned} I\,\alpha\,x\,f\,Z &=_\beta x \\ I\,\alpha\,x\,f\,(S\,y) &=_\beta f(I\,\alpha\,x\,f\,y) \end{aligned}$$

[4 marks]

(*c*) For each natural number $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let $S^n Z$ be the PLC expression given by

$$\begin{aligned} S^0 Z &= Z \\ S^{n+1} Z &= S(S^n Z) \end{aligned}$$

What is the beta-normal form of $S^0 Z$, of $S^1 Z$, of $S^2 Z$, and in general, of $S^n Z$? [4 marks]

(*d*) (*i*) Using part (*b*), or otherwise, show that there is a closed PLC expression A of type $nat \to nat \to nat$ that represents addition of natural numbers, in the sense that $A(S^m Z)(S^n Z) =_\beta S^{m+n} Z$ holds for all $m, n \in \mathbb{N}$. [Hint: recall the primitive recursive definition of addition.] [5 marks]

(*ii*) Show that $M = \lambda y : nat(I\,nat\,Z\,(A\,y))$ (with A is as in part (*i*)) represents multiplication of natural numbers, in the sense that $M(S^m Z)(S^n Z) =_\beta S^{mn} Z$ holds for all $m, n \in \mathbb{N}$. [5 marks]