2010 Paper 6 Question 4

Computation Theory

(a) Define Church's representation of numbers n as λ-terms $\underline{n}$. [3 marks]

(b) What does it mean for a partial function $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ to be λ-*definable*? What is the relationship between λ-definability and computability? [3 marks]

(c) Show that $\texttt{succ}(x_1) = x_1 + 1$ is λ-definable. [4 marks]

(d) Ackermann's function $ack \in \mathbb{N}^2 \to \mathbb{N}$ is a total function of two arguments satisfying

$$\begin{aligned} ack(0, x_2) &= x_2 + 1 \\ ack(x_1 + 1, 0) &= ack(x_1, 1) \\ ack(x_1 + 1, x_2 + 1) &= ack(x_1, ack(x_1 + 1, x_2)). \end{aligned}$$

By considering $\lambda x.\, x\, T\, S$ where $T = \lambda f\, y.\, y\, f\, (f\, \underline{1})$ and S is chosen suitably, prove that Ackermann's function is λ-definable. [10 marks]