2010 Paper 6 Question 1

Complexity Theory

(*a*) Give precise definitions of *polynomial-time reductions* and ***NP**-completeness.* [2 marks each]

(*b*) Prove that for any language L, L is polynomial-time reducible to some problem in **NP** if, and only if, L is in **NP**. [6 marks]

(*c*) In a simple graph $G = (V, E)$, a set of vertices $X \subseteq V$ is said to be a *vertex cover* of G if every edge $e \in E$ has one endpoint in X. A set $X \subseteq V$ is an *independent set* of G if there is no edge between any two vertices in X.

Vertex Cover is defined as the decision problem where, given a graph $G = (V, E)$ and a positive integer k, we are to determine whether G contains a vertex cover with k or *fewer* vertices.

Independent Set is defined as the decision problem where, given a graph $G = (V, E)$ and a positive integer k, we are to determine whether G contains an independent set with k or *more* vertices.

(*i*) Show that a set X is a vertex cover of G if, and only if, its complement $V \setminus X$ is an independent set of G. [2 marks]

(*ii*) Use this to show that Vertex Cover is polynomial-time reducible to Independent Set and *vice versa*. [6 marks]

(*iii*) What can you conclude about the complexity of Vertex Cover? [2 marks]