# 2009 Paper 9 Question 6

## Denotational Semantics

(*a*) (*i*) Define the notion of *contextual equivalence* in PCF. [2 marks]

(*ii*) Consider the following two closed PCF terms of type $nat \to nat \to nat$.

$$\begin{aligned} F \stackrel{\text{def}}{=} \mathbf{fix}\big(\ & \mathbf{fn}\, f : nat \to nat \to nat.\ \mathbf{fn}\, x : nat.\ \mathbf{fn}\ y : nat. \\ & \quad \mathbf{if\ zero}(x)\ \mathbf{then\ 0} \\ & \quad \mathbf{else\ if\ zero}(y)\ \mathbf{then\ 0} \\ & \quad \mathbf{else\ succ}\big(\, f\,(\mathbf{pred}\ x)\,(\mathbf{pred}\ y)\,\big)\ \big) \end{aligned}$$

$$\begin{aligned} G \stackrel{\text{def}}{=} \mathbf{fix}\big(\ & \mathbf{fn}\, g : nat \to nat \to nat.\ \mathbf{fn}\, x : nat.\ \mathbf{fn}\ y : nat. \\ & \quad \mathbf{if\ zero}(y)\ \mathbf{then\ 0} \\ & \quad \mathbf{else\ if\ zero}(x)\ \mathbf{then\ 0} \\ & \quad \mathbf{else\ succ}\big(\, g\,(\mathbf{pred}\ x)\,(\mathbf{pred}\ y)\,\big)\ \big) \end{aligned}$$

Are $F$ and $G$ contextually equivalent? Justify your answer. [5 marks]

(*b*) (*i*) Define a closed PCF term $H : (nat \to nat \to nat) \to nat \to nat \to nat$ such that $[\![\mathbf{fix}(H)]\!] \in \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big)$ satisfies

$$[\![\mathbf{fix}(H)]\!]\,(i)\,(j) \;=\; \max(i,j)$$

for all $i, j \in \mathbb{N}$. [4 marks]

(*ii*) Let

$$S \stackrel{\text{def}}{=} \{\, f \in \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big) \mid f(x)(y) = f(y)(x) \text{ for all } x, y \in \mathbb{N}_\perp \,\}$$

Show that the subset $S \subseteq \big(\mathbb{N}_\perp \to (\mathbb{N}_\perp \to \mathbb{N}_\perp)\big)$ is admissible. [4 marks]

(*iii*) Show that

$$[\![\mathbf{fix}(H)]\!]\,(x)\,(y) \;=\; [\![\mathbf{fix}(H)]\!]\,(y)\,(x)$$

for all $x, y \in \mathbb{N}_\perp$. [5 marks]

[Hint: Use Scott's Fixed-Point Induction Principle.]