2009 Paper 7 Question 8

Denotational Semantics

(*a*) Let D be a poset and let $f : D \to D$ be a monotone function. Give the definition of the *least pre-fixed point*, $\mathit{fix}(f)$, of f. [3 marks]

(*b*) Let D, E be domains and let $p : D \to E$ and $q : E \to D$ be continuous functions.

(*i*) Prove that $\mathit{fix}(q \circ p) \sqsubseteq q\big(\mathit{fix}(p \circ q)\big)$. [4 marks]

(*ii*) Thereby also prove that $p\big(\mathit{fix}(q \circ p)\big) \sqsubseteq \mathit{fix}(p \circ q)$. [4 marks]

Hence,

$$(\star) \qquad \mathit{fix}(q \circ p) = q\big(\mathit{fix}(p \circ q)\big) \quad \text{and} \quad p\big(\mathit{fix}(q \circ p)\big) = \mathit{fix}(p \circ q) \ .$$

(*c*) Let D, E be domains and let $f : (D \times E) \to D$ be a continuous function. Define $f^\dagger : E \to D$ to be the function $f^\dagger \stackrel{\text{def}}{=} \lambda e \in E.\, \mathit{fix}\big(\lambda d \in D.\, f(d, e)\big)$. Show that $f^\dagger$ is continuous. [5 marks]

Analogously, for a continuous function $g : (D \times E) \to E$, let $g^\ddagger : D \to E$ be the function $g^\ddagger \stackrel{\text{def}}{=} \lambda d \in D.\, \mathit{fix}\big(\lambda e \in E.\, g(d, e)\big)$. Then $g^\ddagger$ is continuous.

(*d*) Let D, E be domains and let $f : (D \times E) \to D$ and $g : (D \times E) \to E$ be continuous functions. Define $h : (D \times E) \to (D \times E)$ to be the continuous function $h \stackrel{\text{def}}{=} \lambda (d, e) \in D \times E.\, \big(f(d, e), g(d, e)\big)$.

(*i*) Prove that $\mathit{fix}(h) \sqsubseteq \big(\mathit{fix}(f^\dagger \circ g^\ddagger)\,,\, \mathit{fix}(g^\ddagger \circ f^\dagger)\big)$. [2 marks]

[Hint: Recall ($\star$) above and the pre-fixed point property of $f^\dagger(e)$ and of $g^\ddagger(d)$, for $e = \mathit{fix}(g^\ddagger \circ f^\dagger)$ and $d = \mathit{fix}(f^\dagger \circ g^\ddagger)$.]

(*ii*) Let $\mathit{fix}(h) = (x, y)$.

Prove that $\mathit{fix}(f^\dagger \circ g^\ddagger) \sqsubseteq x$ and $\mathit{fix}(g^\ddagger \circ f^\dagger) \sqsubseteq y$. [2 marks]

[Hint: Recall the pre-fixed point property of $(x, y) = \mathit{fix}(h)$ and the least pre-fixed point property of $f^\dagger(y)$ and of $g^\ddagger(x)$.]

Hence, $\mathit{fix}(h) = \big(\mathit{fix}(f^\dagger \circ g^\ddagger)\,,\, \mathit{fix}(g^\ddagger \circ f^\dagger)\big)$.