2009 Paper 7 Question 11

Information Theory and Coding

(*a*) Let X and Y be discrete random variables over state ensembles $\{x\}$ and $\{y\}$ having probability distributions $p(x)$ and $p(y)$, conditional probability distributions $p(x|y)$ and $p(y|x)$, and joint probability distribution $p(x, y)$. Using only these quantities, provide expressions for each of the following:

(*i*) The joint uncertainty $H(X, Y)$ about both random variables. [2 marks]

(*ii*) The uncertainty $H(X|y = b_j)$ about random variable X once it is known that variable Y has taken on a particular value $y = b_j$. [2 marks]

(*iii*) The average uncertainty $H(X|Y)$ remaining about random variable X when Y is known. [2 marks]

(*iv*) The mutual information $I(X; Y)$ between random variables X and Y. (Your answer can use expressions you have defined above.) [2 marks]

(*v*) The union of $H(X|Y)$, $I(X; Y)$, and $H(Y|X)$. [2 marks]

(*b*) A noisy binary communication channel randomly corrupts bits with probability p, so its channel matrix is:

$$\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

(*i*) If the input bit values $\{0, 1\}$ are equiprobable, what is the mutual information between the input and output for this noisy channel? [2 marks]

(*ii*) What is the *channel capacity* of this noisy channel? [1 mark]

(*iii*) If an error-correcting code were designed for this noisy channel, what would be the maximum possible entropy of an input source for which reliable transmission could still be achieved? Express your answer in terms of p. [1 mark]

(*iv*) Name the theorem by Shannon that is the basis for the result in (*iii*). [1 mark]

(*c*) Explain why the encoding of continuous signals into sequences of coefficients on Gabor wavelets encompasses, as special cases, both the delta function sampling basis and the Fourier Transform basis. Show how one particular parameter determines where a signal representation lies along this continuum that bridges from delta function sampling to the complex exponential. [5 marks]