2008 Paper 8 Question 14

Denotational Semantics

(*a*) Describe the properties a function between two cpos must have to be continuous. [2 marks]

(*b*) Let D_1, D_2 and E be cpos. Prove that a function $h : D_1 \times D_2 \to E$ is continuous if it is continuous in each argument separately. [You may assume standard properties of least upper bounds provided you state them clearly.] [4 marks]

(*c*) Let $\mathbb{O}$ be the cpo with two elements $\bot \sqsubseteq \top$. For a cpo E and $e \in E$, define the function $g_e : E \to \mathbb{O}$ by

$$g_e(x) = \begin{cases} \bot & \text{if } x \sqsubseteq e \\ \top & \text{if } x \not\sqsubseteq e \end{cases}$$

Show g_e is continuous. [4 marks]

(*d*) As an example of the definition in part (*c*) above, let $E = \mathbb{B}_\bot \times \mathbb{B}_\bot$, where $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$, and consider $g_{(\mathit{false},\mathit{false})} : E \to \mathbb{O}$. Show that

$$g_{(\mathit{false},\mathit{false})}(x, y) = \top \text{ iff } x = \mathit{true} \text{ or } y = \mathit{true}$$

[2 marks]

(*e*) Let $f : D \to E$ be a function between cpos D and E. Show

$$f \text{ is continuous iff } \forall e \in E.\quad g_e \circ f \text{ is continuous}$$

[You may assume that the composition of continuous functions is continuous. It is suggested that for the "if" direction of the proof, you argue by contradiction.] [8 marks]