2008 Paper 4 Question 4

Mathematical Methods for Computer Science

(*a*) Consider a *simple random walk*, S_n, defined by $S_0 = a$ and $S_n = S_{n-1} + X_n$ for $n \geq 1$ where the random variables X_i $(i = 1, 2, \ldots)$ are independent and identically distributed with $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p$ for some constant p with $0 \leq p \leq 1$.

(*i*) Find $E(S_n)$ and $Var(S_n)$ in terms of a, n and p. [4 marks]

(*ii*) Use the *central limit theorem* to derive an approximate expression for $P(S_n > k)$ for large n. You may leave your answer expressed in terms of the distribution function $\Phi(x) = P(Z \leq x)$ where Z is a standard Normal random variable with zero mean and unit variance. [6 marks]

(*b*) Consider the *Gambler's ruin problem* defined as in part (*a*) but with the addition of absorbing barriers at 0 and N where N is some positive integer. Derive an expression for the probability of ruin (that is, being absorbed at the zero barrier) when starting at position $S_0 = a$ for each $a = 0, 1, \ldots, N$ in the two cases

(*i*) $p \neq \frac{1}{2}$ [5 marks]

(*ii*) $p = \frac{1}{2}$. [5 marks]