2006 Paper 8 Question 14

Denotational Semantics

Let D be a domain with bottom element $\bot$. Let $h, k : D \to D$ be continuous functions with h strict (so $h(\bot) = \bot$). Let $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$. Define the conditional function

$$\mathit{if} : \mathbb{B}_\bot \times D \times D \to D$$

by $\mathit{if}(b, d, d') = d$ if $b = \mathit{true}$, d' if $b = \mathit{false}$, and $\bot$ otherwise. Let $p : D \to \mathbb{B}_\bot$ be a continuous function.

The function f is the least continuous function from $D \times D$ to D such that

$$\forall x \in D.\ f(x, y) = \mathit{if}(p(x),\ y,\ h(f(k(x), y)))\ .$$

(*a*) State the principle of fixed-point induction. What does it mean for a property to be chain closed? [4 marks]

(*b*) Assume that the property

$$Q(g) \Leftrightarrow_{\mathit{def}} \forall x, y \in D.\ h(g(x, y)) = g(x, h(y))\ ,$$

where g is a continuous function from $D \times D$ to D, is chain closed. Prove $Q(f)$ by fixed-point induction. [7 marks]

(*c*) Let g be a continuous function from a cpo D to a cpo E. Let Y be a chain-closed subset of E. Show that the inverse image $g^{-1}Y$ is a chain-closed subset of D. [4 marks]

(*d*) Exhibit cpos D, E, F and chain-closed subsets $R \subseteq D \times E$ and $S \subseteq E \times F$ such that their relational composition $S \circ R \subseteq D \times F$ is not chain closed. (No proof is required.) [5 marks]