2005 Paper 9 Question 12

Numerical Analysis II

(*a*) Explain the term *positive semi-definite*. If $\mathbf{A}$ is a real square matrix show that $\mathbf{A}^T\mathbf{A}$ is symmetric and positive semi-definite. [3 marks]

(*b*) How is the l_2 *norm* of $\mathbf{A}$ defined? State Schwarz's inequality for the product $\mathbf{Ax}$. [2 marks]

(*c*) Describe briefly the properties of the matrices $\mathbf{U}$, $\mathbf{W}$, $\mathbf{V}$ in the *singular value decomposition* $\mathbf{A} = \mathbf{UWV}^T$. [3 marks]

(*d*) Let $\hat{\mathbf{x}}$ be an approximate solution of $\mathbf{Ax} = \mathbf{b}$, and write $\mathbf{r} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}}$, $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$. Derive a computable estimate of the relative error $\|\mathbf{e}\|/\|\mathbf{x}\|$ in the approximate solution, and show how this may be used with the l_2 norm. [8 marks]

(*e*) Suppose $\mathbf{A}$ is a 7×7 matrix whose singular values are 10^2, 10^{-4}, 10^{-10}, 10^{-16}, 10^{-22}, 10^{-29}, 10^{-56}. Construct the matrix $\mathbf{W}^+$ that you would use (*i*) if *machine epsilon* $\simeq 10^{-15}$, and (*ii*) if *machine epsilon* $\simeq 10^{-30}$. [4 marks]