2005 Paper 8 Question 15

Denotational Semantics

Let D be a domain with bottom element $\perp$. Let $h, k : D \to D$ be continuous functions with h strict (so $h(\perp) = \perp$). Let $\mathbb{B} = \{\mathit{true}, \mathit{false}\}$. Define the conditional function,

$$\text{if} : \mathbb{B}_\perp \times D \times D \to D$$

by $\text{if}(b, d, d') = d$ if $b = \mathit{true}$, d' if $b = \mathit{false}$, and $\perp$ otherwise. Let $p : D \to \mathbb{B}_\perp$ be a continuous function.

The function f is the least continuous function from $D \times D$ to D such that

$$\forall x \in D.\ f(x, y) = \text{if}(p(x),\ y,\ h(f(k(x), y)))\ .$$

(a) State the principle of fixed point induction. What does it mean for a property to be admissible? [4 marks]

(b) Show that

$$\forall b \in \mathbb{B}_\perp, d, d' \in D.\ h(\text{if}(b,\ d,\ d')) = \text{if}(b,\ h(d),\ h(d'))\ .$$

[3 marks]

(c) Prove that the property

$$Q(g) \Leftrightarrow_{def} \forall x, y \in D.\ h(g(x, y)) = g(x, h(y))\ ,$$

where g is a continuous function from $D \times D$ to D, is admissible. [5 marks]

(d) Prove $Q(f)$ by fixed point induction. [8 marks]