2004 Paper 12 Question 9

Numerical Analysis II

(*a*) State a recurrence formula for the sequence of Chebyshev polynomials, $\{T_k(x)\}$, and list these as far as $T_5(x)$. [4 marks]

(*b*) What is the best L_∞ polynomial approximation over $[-1, 1]$ to x^k using polynomials of lower degree, and what is its degree? Use this property to explain the method of economisation of a Taylor series. How can the error in one economisation step be estimated? [7 marks]

(*c*) It is required to approximate the function $f(x) = \lim_{k \to \infty} P_k(x)$ over $[-1, 1]$ with an absolute accuracy of 2 decimal places, where

$$P_k(x) = \sum_{n=1}^{k} \frac{x^n}{n\, n!} \; .$$

As this series converges faster than e^x, a good estimate of the error $||f(x) - P_k(x)||_\infty$ in the truncated Taylor series is given by evaluating the next term

$$\frac{x^{k+1}}{(k+1)(k+1)!}$$

at $x = 1$. Use the method of economisation to find a polynomial approximation of the required accuracy. [9 marks]