1999 Paper 10 Question 13

Continuous Mathematics

Many important problems in mathematical modelling and scientific computing require the use of complex variables. Unfortunately, popular programming languages like C do not have a complex variable type, and so we must construct them from floating-point types. Assuming that the quantities a, b, c, d are all real numbers and $i = \sqrt{-1}$, resolve the following expressions, or explain the following operations, involving complex variables $\mathcal{Z}_1 = a + ib$ and $\mathcal{Z}_2 = c + id$:

(a) Let $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$. What is the real part of $\mathcal{Z}_3$, and what is its imaginary part? [2 marks]

(b) What is $\|\mathcal{Z}_1\|$, the modulus of $\mathcal{Z}_1$, and what is $\|\mathcal{Z}_3\|$, the modulus of $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$? [2 marks]

(c) What is $\angle\mathcal{Z}_2$, the angle of complex variable $\mathcal{Z}_2$? [2 marks]

(d) Express $\mathcal{Z}_1$ in complex polar form, not using the quantities a or b but rather the modulus $\|\mathcal{Z}_1\|$ and angle $\angle\mathcal{Z}_1$. [2 marks]

(e) Suppose that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ both have a modulus of 1. Explain, with the aid of a diagram, how their product $\mathcal{Z}_3 = \mathcal{Z}_1\mathcal{Z}_2$ amounts to a rotation in the complex plane. Why is the multiplication of these complex variables reduced now to addition? Without using the quantities a, b, c, d, what is the value of $\|\mathcal{Z}_3\|$? [4 marks]

(f) Suppose that in complex polar form, $\mathcal{Z} = \exp(2\pi i/5)$. What do you get if $\mathcal{Z}$ is multiplied by itself 5 times? Give the simplest possible answer that you can. [2 marks]

(g) Consider the complex exponential function $f(x) = \exp(2\pi i\omega x)$. What function is its real part? What function is its imaginary part? [2 marks]

(h) If the above function $f(x)$ passes through a linear system, i.e. is operated upon by any conceivable linear differential or integral operator, what is the most dramatic way in which $f(x)$ can possibly be affected? [4 marks]