1997 Paper 7 Question 5

Denotational Semantics

Suppose that for each domain D we wish to have a function $Y_D : (D \to D) \to D$ (where as usual, $D \to D$ denotes the domain of continuous functions from D to itself). Consider the following two properties:

(a) For all continuous functions $f : D \to D$, $Y_D(f) = f(Y_D(f))$.

(b) For all domains E and for all continuous functions $f : D \to D$, $g : E \to E$ and $h : D \to E$, with h strict, if $h \circ f = g \circ h$ then $Y_E(g) = h(Y_D(f))$.

Prove that there is such a family of functions Y_D satisfying (a) and (b). Any facts that you use about the existence of fixed points must be proved. [10 marks]

Let $\Omega = \{d_n \mid n \geqslant 0\} \cup \{d_\omega\}$ be the domain consisting of a countably infinite chain together with its least upper bound: thus the non-trivial instances of the partial order relation on Ω are $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_\omega$. Let $s : \Omega \to \Omega$ be the continuous function that sends each d_n to d_{n+1} and sends d_ω to itself. For each continuous function $g : E \to E$ on a domain E, show that there is a strict continuous function $h : \Omega \to E$ satisfying $h \circ s = g \circ h$. Hence, or otherwise, deduce that there is only one family of functions Y_D satisfying properties (a) and (b). [10 marks]