1997 Paper 2 Question 4

Probability

A gate in a communications network is always in one of two states, *open* or *closed*. At each clock pulse it may change state according to the following rules:

- If it is *open* it remains open with probability $1-\alpha$ and changes to *closed* with probability α.
- If it is *closed* it remains closed with probability $1-\beta$ and changes to *open* with probability β.

It may be assumed that $0 < \alpha < 1$ and that $0 < \beta < 1$. Let u_n be the probability that the gate is *closed* just after the nth clock pulse. Derive a difference equation for u_n and justify your derivation. [4 marks]

Let u_0 be the probability that the gate is *closed* initially. Solve your difference equation so as to give u_n in terms of α, β, u_0 and n. [7 marks]

Determine an expression for u_n in the limit as $n \to \infty$ and explain informally why this does not depend on u_0. [3 marks]

Find u_n in each of the four extreme cases: $\alpha = 0$, $\beta = 0$; $\alpha = 0$, $\beta = 1$; $\alpha = 1$, $\beta = 0$; and $\alpha = 1$, $\beta = 1$. Explain the operation of the system in each case. [6 marks]