1996 Paper 12 Question 14

Numerical Analysis II

In Peano's theorem, if a quadrature rule integrates polynomials of degree N exactly over an interval $[a, b]$, then the error in integrating $f \in C^{N+1}[a, b]$ is conventionally expressed as

$$E(f) = \int_a^b f^{(N+1)}(t) K(t)\, dt$$

where

$$K(t) = \frac{1}{N!} E_x[(x-t)_+^N].$$

Explain the notation $(x-t)_+^N$ and E_x. [3 marks]

It follows directly from Taylor's theorem that

$$E(f) = \frac{1}{N!} E_x \left\{ \int_a^x f^{(N+1)}(t)(x-t)^N\, dt \right\}.$$

Explain clearly, in simple stages, how to complete the proof of Peano's theorem. [8 marks]

For the mid-point rule, what is N? [1 mark]

If $K(t)$ does not change sign in $[a, b]$ then

$$E(f) = \frac{f^{(N+1)}(\xi)}{(N+1)!} E(x^{N+1})$$

for some $\xi \in (a, b)$. Use this result to simplify

$$E(f) = \int_{-1}^1 f(x)\, dx - 2f(0)$$ [8 marks]