1993 Paper 9 Question 11

Types

Describe the relation of β*-reduction* between expressions in the second order lambda calculus $\lambda 2$. Explain the *Church–Rosser* and *strong normalisation* properties of this relation. How do they lead to a procedure for deciding whether two closed, typable $\lambda 2$ expressions are β-convertible? [8 marks]

If α is a type variable and σ is a $\lambda 2$ type, the 'existential type' $\exists\alpha.\sigma$ is defined to be $\forall\beta.((\forall\alpha.\sigma\to\beta)\to\beta)$, where β is a new type variable not occurring in σ. Show that there is a closed $\lambda 2$ expression, *Pair*, of type $\forall\alpha.(\sigma\to\exists\alpha.\sigma)$. [5 marks]

Given valid $\lambda 2$ type assertions

$$\begin{aligned} \Gamma &\vdash E : \exists\alpha.\sigma \\ \Gamma, x : \sigma &\vdash N : \sigma' \end{aligned}$$

where α and β do not occur free in Γ or σ', show how to construct a $\lambda 2$ expression $\mathit{Case}(E, \alpha, x, N)$ such that

(a) $\Gamma \vdash \mathit{Case}(E, \alpha, x, N) : \sigma'$, and

(b) if $\Gamma \vdash M : (\alpha \mapsto \tau)\sigma$ where τ is a type not containing free occurrences of β, then

$$\mathit{Case}(\mathit{Pair}_\tau M, \alpha, x, N) \quad \beta\text{-reduces to} \quad (x \mapsto M)(\alpha \mapsto \tau)N.$$

(Here $(\alpha \mapsto \tau)$ denotes the operation of substituting τ for the type variable α and $(x \mapsto M)$ denotes the operation of substituting the expression M for the identifier x.)

[7 marks]