Technical Report

UCAM-CL-TR-225
ISSN 1476-2986

Number 225

UNIVERSITY OF CAMBRIDGE
Computer Laboratory

Categorical multirelations, linear logic and petri nets (draft)

Valeria de Paiva

May 1991

15 JJ Thomson Avenue
Cambridge CB3 0FD
United Kingdom
phone +44 1223 763500

http://www.cl.cam.ac.uk/

Technical reports published by the University of Cambridge
Computer Laboratory are freely available via the Internet:

http://www.cl.cam.ac.uk/TechReports/

ISSN 1476-2986

Categorical Multirelations, Linear Logic and Petri Nets (DRAFT)

Valeria de Paiva

May 31, 1991

Abstract

This note presents a categorical treatment of multirelations, which is, in a loose sense a generalisation of both our previous work on the categories **GC**, [dP'89] and of Chu's construction $A_N\mathbf{C}$ [Barr'79]. The main motivation for writing this note was the utilisation of the category **GC** by Brown and Gurr [BG90] to model Petri Nets. We wanted to extend their work to deal with multirelations, as Petri Nets are usually modelled using multirelations *pre* and *post*. That proved easy enough and people interested mainly in concurrency theory should refer to our joint work [BGdP'91], this note deals with the mathematics underlying [BGdP'91]. The upshot of this work is that we build a model of Intuitionistic Linear Logic (without modalities) over any symmetric monoidal category **C** with a distinguished object $(N, \leq, \circ, e \multimap)$ - a closed poset. Moreover, if the category **C** is cartesian closed with free monoids, we build a model of Intuitionistic Linear Logic with a ***non-trivial*** modality '!' over it.

Introduction

This note extends the treatment of relations in the category **GC** - which is an interesting model of (Full Intuitionistic) Linear Logic [dP89] see sequent presentation in the appendix1 - to multirelations over a category **C** where **C** is a category with a distinguished (ordered) object $(N, \leq)$. In particular we discuss the case of multirelations [Wins88] in the category **Sets**, where $(N, \leq)$ is the set of natural numbers $\mathbf{N}$ with its usual ordering.

The main motivation for writing this note was the utilisation of the category **GC** by Brown and Gurr [BG90] to model Petri Nets. The idea of using category theory to model Petri Nets originates with Glynn Winskel, who attributes some of the insights to Mike Fourman. Brown and Gurr following Winskel's lead and also Girard's dictum that Linear Logic ought to relate nicely to Concurrency, used the category **GC** to model Petri Nets, but as **GC** dealt with relations, they could only account for particular Petri Nets, called in some of the literature ***elementary Petri Nets.*** Elementary Petri nets are nets where the relations *pre* and *post* have multiplicities restricted to $0 - 1$. It seemed to us that it should be easy to extend the treatment in [BG90] to modelling Petri nets with multiplicities; that turned to be the case and people interested mainly in concurrency theory should consult [BGdP'91]. This note deals with the mathematics underlying [BGdP'91], which could not be all explained in that paper.

The construction described here can also be seen as a common generalisation of the constructions of the category **GC** cf.[dP89] and of $GAME_K$ of Yves Lafont [YL'88], [LS'91]. Note that, as the category $GAME_K$ can be seen as a special case of Chu's construction $A_N\mathbf{C}$ - cf. the appendix of ****-Autonomous Categories*** [Bar79] - this note compares **GC** and Chu's category.

In the first section we present our generalisation of Chu's construction, which we call the category $M_N\mathbf{C}$ and compare the two constructions. We also state a proposition interesting from the abstract viewpoint, but not explicitly used anywhere in the paper and whose proof, by Dominic Verity, would make this note even longer. This proposition shows that one of Chu's main results, that his category was enriched over the base category **C**, is also true of our construction $M_N\mathbf{C}$. In the second section, to motivate the richer structure on the category $M_N\mathbf{C}$, we restrict ourselves to the case where $\mathbf{C}$ is the category **Sets** and N is the set of the natural numbers with its usual order. That is the interesting case for Petri Nets applications. In the third section we describe the multiplicative and additive structures of $M_N\mathbf{C}$ in the general case. In the fourth and longest section we discuss Linear Logic modality '!' for $M_N\mathbf{C}$ under the strong assumption that $\mathbf{C}$ is cartesian closed with free commutative monoids. This section is a straightforward generalisation of our previous results for **GC**, but the calculations are slightly more complicated and quite lengthy. Finally, in the last section we describe some of the possible generalisations of $M_N\mathbf{C}$ under investigation and their possible applications.

This work was first presented at the Edinburgh Workshop in Concurrency, Petri Nets and Linear Logic in April 1990 and subsequently at the CLICS review meeting in Paris, September 90. Many thanks to these audiences, in particular to Martin Hyland, Jean-Yves Girard, Carolyn Brown, Doug Gurr, Harold Schllinx, Andy Pitts and Dominic Verity. Thanks also to Peter Dybjer, who made me write about the relationship between $M_N\mathbf{C}$ and $GAME_K$.

1 Chu's Construction Revisited

In this section we define a category $M_N\mathbf{C}$ and compare it to Chu's original construction in [Barr'79]. To construct $M_N\mathbf{C}$ we assume that $\mathbf{C}$ is a symmetric monoidal closed category and N is a distinguished object of $\mathbf{C}$ equipped with a partial order.

The objects of the category $M_N\mathbf{C}$ are triples (U, X, α) where U and X are objects of $\mathbf{C}$ and $U \otimes X \xrightarrow{\alpha} N$ is a morphism in $\mathbf{C}$. We write this triple as $(U \overset{\alpha}{\nleftarrow} X)$ and call it the object A.

To define the morphisms in $M_N\mathbf{C}$, first note that the order in the object $(N, \leq)$ of $\mathbf{C}$ induces an order on the homset $\mathbf{C}(U, N)$ given by

$$\text{for} \quad f, g\colon U \to N \quad f \leq g \qquad \text{iff} \qquad \forall u \in U \quad f(u) \leq g(u)$$

Then for objects $A = (U \overset{\alpha}{\nleftarrow} X)$ and $B = (V \overset{\beta}{\nleftarrow} Y)$ in $M_N\mathbf{C}$, say that a morphism from A to B corresponds to a pair of morphisms in $\mathbf{C}$, (f, F), $f : U \to V$ and $F : Y \to X$ such that, in the following diagram,

$$\begin{tikzcd}
U \otimes Y \arrow[r, "U \otimes F"] \arrow[d, "f \otimes Y"'] & U \otimes X \arrow[d, "\alpha"] \\
V \otimes Y \arrow[r, "\beta"'] & N
\end{tikzcd}$$

we have $\alpha \circ (U \otimes F) \leq (f \otimes Y) \circ \beta$ as morphisms in $\mathbf{C}(U \otimes Y, N)$.

Diagramatically we have:

$$\begin{tikzcd}
U \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow"] & X \arrow[l, "\alpha"'] \\
V & Y \arrow[l, "\beta"] \arrow[u, "F"']
\end{tikzcd} \qquad \forall u \otimes y \in U \otimes Y \quad \alpha(u \otimes Fy) \leq \beta(fu \otimes y)$$

The data above can be collected in the following definition.

Definition 1 *Given a symmetric monoidal closed category* $\mathbf{C}$ *with a distinguished object* $(N, \leq)$ *the category* $M_N\mathbf{C}$ *consists of:*

- *objects are triples* (U, X, α) *written as* $(U \xleftarrow{\alpha} X)$, *where* $U \otimes X \xrightarrow{\alpha} N$ *is a morphism in* $\mathbf{C}$;
- *morphisms are pairs of maps* (f, F) *in* $\mathbf{C}$, $f: U \to V$ *and* $F: Y \to X$, *such that in the following diagram*

$$\begin{tikzcd}
U \otimes Y \arrow[r, "U \otimes F"] \arrow[d, "f \otimes Y"'] & U \otimes X \arrow[d, "\alpha"] \\
V \otimes Y \arrow[r, "\beta"'] & N
\end{tikzcd}$$

we have $\alpha \circ (U \otimes F) \leq (f \otimes Y) \circ \beta$ *as morphisms in* $\mathbf{C}(U \otimes Y, N)$.

Identities in $M_N\mathbf{C}$ are identities of $\mathbf{C}$ in each coordinate, composition is given by composition in each coordinate and associativity comes from the associativity in $\mathbf{C}$. Thus we have the following proposition.

Proposition 1 *The description above defines a category* $M_N\mathbf{C}$.

The only thing to check is composition of morphisms $(f, F): A \to B$ and $(g, G): B \to C$, which is easily done using the diagram:

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l, "\alpha"'] \\
V \arrow[d, "g"'] & Y \arrow[l, "\beta"'] \arrow[u, "F"'] \\
W & Z \arrow[l, "\gamma"'] \arrow[u, "G"']
\end{tikzcd}
\qquad
\begin{array}{l}
\forall u \otimes y \in U \otimes Y \ \alpha(u \otimes Fy) \leq \beta(fu \otimes y) \\[3em]
\forall v \otimes z \in V \otimes Z \ \beta(v \otimes Gz) \leq \gamma(gv \otimes z)
\end{array}$$

We clearly have composite morphisms $gf: U \to W$ and $FG: Z \to X$ in $\mathbf{C}$. Now for $u \in U$ and $z \in Z$, we have $\alpha(u \otimes FGz) \leq \beta(fu \otimes Gz) \leq \gamma(gfu \otimes z)$, which shows that (gf, FG) is a morphism in $M_N\mathbf{C}$. □

Note that composition really corresponds to the following diagram:

$$\begin{tikzcd}
U \otimes Z \arrow[r, "U \otimes G"] \arrow[d, "f \otimes Z"'] \arrow[dr, phantom, "="] & U \otimes Y \arrow[r, "U \otimes F"] \arrow[d, "f \otimes Y"] \arrow[dr, phantom, "\geq"] & U \otimes X \arrow[dd, "\alpha"] \\
V \otimes Z \arrow[r, "V \otimes G"'] \arrow[d, "g \otimes Z"'] \arrow[dr, phantom, "\geq"] & V \otimes Y \arrow[dr, "\beta"] & \\
W \otimes Z \arrow[rr, "\gamma"'] & & N
\end{tikzcd}$$

1.1 First Comparison

It is clear, for those who know Chu's construction,[Barr'79], that the category $M_N\mathbf{C}$ is - in one sense - an easy generalisation of it. To keep this note self-contained we recall the definition of Chu's category, which he calls A_N - A for autonomous - . There is a small clash of notation, as Chu uses X for the distinguished object, which we call N and he does not mention the base category $\mathbf{C}$, whereas we want to have it explicit. Thus we write Chu's category as $A_N\mathbf{C}$. The category $A_N\mathbf{C}$ can be constructed over any symmetric monoidal closed category $\mathbf{C}$ with pullbacks, see definition below.

Definition 2 *Given a symmetric monoidal closed category* $\mathbf{C}$ *with pulbacks and a chosen object* N*, the category* $A_N\mathbf{C}$ *consists of*

- *Objects are triples* (U, X, α)*, where* $U \otimes X \xrightarrow{\alpha} N$ *is a morphism in* $\mathbf{C}$*;*
- *Morphisms in* $\mathbf{A}_N$ *from an object* $U \otimes X \xrightarrow{\alpha} N$ *to an object* $V \otimes Y \xrightarrow{\beta} N$ *are pairs of morphisms in* $\mathbf{C}$, (f, F) $f: U \to V$ *and* $F: Y \to X$ *such that the following diagram commutes.*

$$\begin{tikzcd}
U \otimes Y \arrow[r, "U \otimes F"] \arrow[d, "f \otimes Y"'] & U \otimes X \arrow[d, "\alpha"] \\
V \otimes Y \arrow[r, "\beta"'] & N
\end{tikzcd}$$

The commutativity of the diagram above means that for all $u \otimes y$ in $U \otimes Y$ one has $\alpha(u \otimes Fy) = \beta(fu \otimes y)$ in $A_N\mathbf{C}$, when we only ask in $M_N\mathbf{C}$ that $\alpha(u \otimes Fy) \leq \beta(fu \otimes y)$, so $M_N\mathbf{C}$ is a generalisation of $A_N\mathbf{C}$.

On the other hand, $M_N\mathbf{C}$ 'looks like' a special case of Chu's construction, as we need a partial order on the object N, whereas for Chu's construction any object in $\mathbf{C}$ will do. But one could say that Chu is using the trivial partial order in N, the one which says $n \leq m$ iff $n = m$.

To compare Chu's construction with $GAME_K$ note that the category **Sets** is trivially symmetric monoidal closed - as it is cartesian closed. Also instead of writing N for the distinguished object write K to make explicit the analogy with vector spaces. Thus $GAME_K$ is $\mathbf{A}_K\mathbf{Sets}$ in our notation, but note that $GAME_K$ arises from a more restricted case in Linear Algebra, where U is a vector space over a field K and X is the collection of linear functionals on U, U^*.

To compare $M_N\mathbf{C}$ with $\mathbf{GC}$, first recall that we can write usual relations on **Sets** either as a subset of a product $A \rightarrowtail U \times X$ or as a function into 2, $U \times X \xrightarrow{\alpha} 2$. Thus to talk about relations in a general categorical set-up one can use either subobjects of a (possibly tensor) product $A \rightarrowtail U \otimes X$ - call it the 'subobject' approach - or maps of the form $U \otimes X \xrightarrow{\alpha} 2$ if your category has an object 2 - call it the 'span' approach. In our previous work with $\mathbf{GC}$ [dP89] we used the subobject approach. Thus objects in $\mathbf{GC}$ are (classes of equivalences of) monics $A \overset{\alpha}{\rightarrowtail} U \times X$ and morphisms are pairs of maps in $\mathbf{C}$ satisfying the same conditions as the ones for $M_N\mathbf{C}$.

Each approach to 'categorical relations' has advantages and disadvantages. For instance, using the 'subobject' approach one can talk about 'decidable' and 'undecidable' relations, if your ambiance category $\mathbf{C}$ has enough structure. This notion of 'decidability' comes from topos theory, [PTJ page 162]. There is an interesting problem to look at if the constructions of this paper are carried inside a realisability universe, for instance the effective topos, where other (recursion-theoretic) notions of decidability could be used.

On the other hand, using the 'span' approach and looking for instance at $M_2\mathbf{C}$ - where $\mathbf{C}$ is a ccc with coproducts - all your objects are decidable. Given a relation $\alpha: U \times X \to 2$, you can always produce $\bar{\alpha}: U \times X \to 2$, saying that $u\bar{\alpha}x$ iff it is not the case that $u\alpha x$.

Thus we can draw the following diagram

$$\begin{tikzcd}
M_N\mathbf{C} \arrow[d] \arrow[dr] & \\
A_N\mathbf{C} \arrow[d] & \mathbf{GC} \\
GAME_K &
\end{tikzcd}$$

and summarize as follows:

- The category $GAME_K$ is $A_N\mathbf{C}$, where $A_N(-)$ denotes Chu's construction; N is the set K and $\mathbf{C}$ is the category of **Sets**.
- The categories $A_N\mathbf{C}$ and $M_N\mathbf{C}$ have the same objects, but there is an inclusion of the set of morphisms. The morphisms of $A_N\mathbf{C}$ are contained in the morphims of $M_N\mathbf{C}$, thus for any pair of objects A and B
$$A_N\mathbf{C}(A,B) \subseteq M_N\mathbf{C}(A,B)$$
- The morphisms of $M_N\mathbf{C}$ and $\mathbf{GC}$ are the same for any two objects A and B,
$$M_N\mathbf{C}(A,B) = \mathbf{GC}(A,B)$$
but the objects of $\mathbf{GC}$ can only be compared with the objects of $M_2\mathbf{C}$. For objects in $M_2\mathbf{C}$ we have $|M_2\mathbf{C}| \subseteq |\mathbf{GC}|$, as all objects in $M_2\mathbf{C}$ are decidable.
- Both categories $A_N\mathbf{C}$ and $M_N\mathbf{C}$ are symmetric monoidal closed categories with finite products and coproducts. The symmetric monoidal closed structures are very similar, but different - more about that in section 3.

We now state a proposition, not necessary for the rest of this note, whose proof can be found in [Ver91]. The reason for mentioning the proposition is that its analogue for $A_N\mathbf{C}$ was one of Chu's main results in [Barr'79]. Note, however, that to prove this proposition, we assume that $\mathbf{C}$ has *pullbacks* as well as being a symmetric monoidal closed category with finite products.

Proposition 2 *The category $M_N\mathbf{C}$ is enriched over the category $\mathbf{C}$.*

The next step is to define more structure in $M_N\mathbf{C}$. Given objects A and B respectively, $(U \overset{\alpha}{\leftrightarrow} X)$ and $(V \overset{\beta}{\leftrightarrow} Y)$ in $M_N\mathbf{C}$, we want to define an internal hom $[A,B]$. The object $[A,B]$ should look like $(V^U \times X^Y \overset{\alpha \multimap \beta}{\leftrightarrow} U \times Y)$, but the problem is to define a map

$$\alpha \multimap \beta \colon V^U \times X^Y \times U \times Y \longrightarrow N$$

with good properties. By good properties it is meant mean that we should be able to define an adjoint tensor product to the internal hom. That can be difficult in the general case, but if $\mathbf{C}$ is Sets and N really is $\mathbb{N}$ the set of natural numbers it is easy. We start with the easy case in the next section and then do the more general one in section 3.

2 The Category MSets

If we consider the construction of $M_N\mathbf{C}$ where $\mathbf{C}$ is the category of sets and usual maps **Sets** and N is the set of natural numbers $\mathbf{N}$, then our objects $(U \xleftarrow{\alpha} X)$ correspond to multirelations $U \times X \xrightarrow{\alpha} \mathbf{N}$ cf. [Wins]. Recall that **Sets** is not only cartesian closed with coproducts, but a topos.

A morphism $(f, F): A \to B$ in $M_{\mathbf{N}}\mathbf{Sets}$ corresponds to a condition on multirelations α and β saying that

$$\forall u \in U, \forall y \in Y \qquad \alpha(u, Fy) \leq \beta(fu, y)$$

as natural numbers, or equivalently that $\forall u \in U, \forall y \in Y \; -\alpha(u, Fy) + \beta(fu, y) \geq 0$.

Moreover the set of natural numbers $\mathbf{N}$ has some extra structure, apart from its usual order, that allows us to define a symmetric monoidal closed structure in $M_{\mathbf{N}}\mathbf{Sets}$. For a start we can add natural numbers, so we can define a tensor product in $M_{\mathbf{N}}\mathbf{Sets}$, which we call from now on **MSets**. Before defining the tensor product, we summarize the discussion above in a definition.

Definition 3 *The category* MSets *consists of:*

- *Objects are triples (U, X, α) written as $(U \xleftarrow{\alpha} X)$, where $U \times X \xrightarrow{\alpha} \mathbf{N}$ is a function in* **Sets**, *that is a multirelation.*

- *Morphisms in* **MSets** *from an object $U \times X \xrightarrow{\alpha} \mathbf{N}$ or $(U \xleftarrow{\alpha} X)$ to an object $V \times Y \xrightarrow{\beta} \mathbf{N}$ or $(V \xleftarrow{\beta} Y)$ are pairs of morphisms in* **Sets**, *(f, F) where $f: U \to V$ and $F: Y \to X$ are such that*

$$\begin{tikzcd} U \arrow[d, "f"'] & X \arrow[l, "\alpha"'] \\ V \arrow[ur, phantom, "\Downarrow"] & Y \arrow[l, "\beta"] \arrow[u, "F"'] \end{tikzcd}$$

That means that $\forall u \in U, \forall y \in Y \; -\alpha(u, Fy) + \beta(fu, y) \geq 0$.

This definition is just an instance of the definition of $M_N\mathbf{C}$ in the first section, so we have a category. Also recall that both $\alpha(u, Fy)$ and $\beta(fu, y)$ are natural numbers, hence the condition above makes sense. We use the usual addition of natural numbers to define a tensor product in **MSets**:

Definition 4 *Given two objects $(U \xleftarrow{\alpha} X)$ and $(V \xleftarrow{\beta} Y)$ in* **MSets** *we define $A \otimes B$ their tensor product as the following object:*

$$A \otimes B = (U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U)$$

where the multirelation "$\alpha \otimes \beta$" is given by $\alpha \otimes \beta(u, v, f, g) = \alpha(u, fv) + \beta(v, gu)$.

To give a formal definition of $\alpha \otimes \beta$, consider the composition:

$$U \times V \times X^V \times Y^U \xrightarrow{\text{“}(ev, ev)\text{”}} U \times V \times X \times Y \xrightarrow{\alpha \times \beta} N \times N \xrightarrow{+} N$$

Note that we are using the cartesian closed structure of **Sets** - to write X^V and Y^U - and the monoidal structure '$+$' of $\mathbf{N}$ to define $\alpha \otimes \beta$.

This operation clearly defines a bifunctor, which is a tensor product. Associativity and commutativity are straightforward and the object $I = (1 \overset{0}{\leftarrow} 1)$ - where the multirelation $1 \times 1 \xrightarrow{0} \mathbf{N}$ 'picks' the 0 of N - is the identity for this tensor product.

Actually this odd-looking tensor product, analogous to the one in **GC**, is the right one to prove monoidal-closedness of **MSets** with respect to a (reasonably) intuitive internal-hom, which we proceed to define.

Definition 5 *Given two objects* $(U \overset{\alpha}{\leftarrow} X)$ *and* $(V \overset{\beta}{\leftarrow} Y)$ *in* **MSets** *we define* $[A, B]$ *their internal-hom as the object,*

$$[A, B] = (V^U \times X^Y \xleftarrow{\alpha \multimap \beta} U \times Y).$$

The multirelation "$(\alpha \multimap \beta)$*" is given by* $(\alpha \multimap \beta)(f, F, u, y) = \dot{-}\alpha(u, Fy) + \beta(fu, y)$*, where the dotted subtraction is truncated subtraction, that is* $\dot{-}\alpha + \beta = \beta - \alpha$ *if* $\alpha \leq \beta$ *and* 0 *otherwise.*

The truncated subtraction in the definition above is very intuitive after reading Lawvere's *"Metric Spaces, Generalised Logic and Closed Categories"*[Law], where the same kind of construction is done using the positive real numbers instead of the natural numbers.

Proposition 3 *The construction above defines a bifunctor* $[-,-]\colon \mathbf{MSets}^{op} \times \mathbf{MSets} \to \mathbf{MSets}$.

Having defined an internal hom and a tensor product we have the obvious:

Theorem 1 *The category* **MSets** *is a symmetric monoidal closed category with respect to the tensor product* $\otimes$ *and the internal-hom* $[-,-]$ *defined above.*

The proof is simple, one has to verify the natural isomorphism

$$Hom_{\mathbf{MSets}}(A \otimes B, C) \cong Hom_{\mathbf{MSets}}(A, [B, C])$$

This can be done by looking at the diagrams

$$\begin{tikzcd}
U \times V \arrow[d, "f"'] & X^V \times Y^U \arrow[l, "\alpha \otimes \beta"'] \\
W & Z \arrow[l, "\gamma"] \arrow[u, "(F_1, F_2)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, "(f, F_2)"] & X \arrow[l, "\alpha"'] \\
W^V \times Y^Z & V \times Z \arrow[l, "\beta \multimap \gamma"'] \arrow[u, "F_1"']
\end{tikzcd}$$

and calculating the sums. If the morphism $(f, (F_1, F_2))$ is in $Hom_{\mathbf{MSets}}(A \otimes B, C)$, then we know $-(\alpha \otimes \beta) + \gamma \geq 0$, which means $-\alpha(u, F_1 zv) - \beta(v, F_2 zu) + \gamma(f(u, v), z) \geq 0$. But to show that the corresponding morphism $(\langle f, F_2 \rangle, F_1)$ is in $Hom_{\mathbf{MSets}}(A, [B, C])$ we have to show $-\alpha + (\beta \multimap \gamma) \geq 0$, which corresponds to $-\alpha(u, F_1(v, z)) + [-\beta(v, F_2 uz) + \gamma(fuv, z)] \geq 0$, which we know, if transposing is allowed. □

In the next section we generalise the constructions of this section to categories other than **Sets** and to N's other than the set of natural numbers.

3 Structure on $M_N C$

The categorically-minded reader may have noticed that we used an "adjointness situation" in $\mathbf{N}$ to define the symmetric monoidal closed structure in **MSets**. That is we have used the facts that in $\mathbf{N}$ we can say $n \leq m$, also $n + m \in \mathbf{N}$ and $\dot{-}n + m \in \mathbf{N}$ - that is '+' and '$\dot{-}$' are bifunctors and there is an adjunction:

$$\dot{-}(m+n) + p \geq 0 \text{ iff } \dot{-}m + (\dot{-}n + p) \geq 0.$$

Thus to generalise the construction of **MSets** we first define "a symmetric monoidal closed poset" $(N, \leq, \circ, \multimap, e)$, then we show how the closed structure of N allows us to define a symmetric monoidal closed structure in $M_N\mathbf{C}$, if $\mathbf{C}$ is symmetric monoidal closed with products.

We should mention that Flagg has, independently, the same definition of a (symmetric monoidal) closed poset in [Fla'90], but he really considers *integral closed posets*, the ones where the identity for the monoidal structure 'e' is also the identity for an extra 'additive' structure, exactly the condition we want to avoid. We give our definition in two easy steps.

Definition 6 *An* ordered monoid $(N, \leq, \circ, e)$ *is a poset* $(N, \leq)$ *with a given compatible symmetric monoidal structure* $(N, \circ, e)$. *The structures are compatible in the sense that, if* $a \leq b$, *we have* $a \circ c \leq b \circ c$, *for all* c *in* N.

These are called *ordered monoids* in Concurrency Theory, but could as well be called *and posets* as in [HdP'91]. Note that to be very precise we should call the monoids above, *symmetric* ordered monoids, see [dP'91] for a slightly more general notion.

Definition 7 *Suppose* $(N, \leq, \circ, e)$ *is an ordered monoid and* $a, b \in N$. *If there exists a largest* $x \in N$ *such that* $a \circ x \leq b$ *then this element is denoted* $a \multimap b$ *and it is called the* relative pseudocomplement of a wrt b. *A* closed poset *is an ordered monoid* $(N, \leq, \circ, e)$ *such that* $a \multimap b$ *exists for all* a *and* b *in* N.

Since we defined a closed poset to be a restriction of the notion of a symmetric monoidal closed category to the category of **Posets**, we have an obvious proposition:

Proposition 4 *A closed poset* $(N, \leq, \circ, e, \multimap)$ *has the folowing properties:*

1. $a \circ b \leq c$ *iff* $a \leq b \multimap c$
2. *If* $a \leq b$, *then for any* c *in* N, $c \multimap a \leq c \multimap b$ *and* $b \multimap c \leq a \multimap c$;
3. *As 'e' is the identity for '$\circ$'* $a \circ e = a \leq a$ *implies* $e \leq a \multimap a$ *for any* a *in* N.

Note that set of the natural numbers with its usual ordering and operations - addition and truncated subtraction - defined in section 2 is a closed poset $\langle \mathbf{N}, \leq, +, 0, \dot{-} \rangle$. For other interesting examples of closed posets see [Flagg].

Having done the first generalisation - to consider a closed poset, instead of the set of natural numbers - we now proceed to generalise the category **Sets** to any symmetric monoidal closed category $\mathbf{C}$ with finite products. Thus, suppose that $\mathbf{C}$ is a symmmetric monoidal closed category with finite products and that $(N, \leq, \circ, e, \multimap)$ is a closed poset as above. Write $[-,-]$ for the internal hom and $\otimes$ for (its adjoint) tensor product in $\mathbf{C}$, as well as $\times$ for the cartesian product. Then we can construct the category $M_N\mathbf{C}$ as in section 1 and one of the possible symmetric monoidal structures of $M_N\mathbf{C}$ is given by:

Definition 8 *Given two objects* $A = (U \overset{\alpha}{\nleftarrow} X)$ *and* $B = (V \overset{\beta}{\nleftarrow} Y)$ *in* $M_N\mathbf{C}$ *we define* $A \otimes_M B$ *their tensor product as follows:*

$$A \otimes_M B = (U \otimes V \overset{(\alpha \otimes \beta)_M}{\longleftarrow\!\!\!\!\!\!+\!\!\!\!\!\!\longrightarrow} [V, X] \times [U, Y])$$

The morphism "$(\alpha \otimes \beta)_M$" intuitively says $(\alpha \otimes \beta)_M(u \otimes v, (f, g)) = \alpha(u \otimes fv) \circ \beta(v \otimes gu)$, *where* $\circ$ *is the monoidal structure in* $(N, \leq, \circ, e, \multimap)$.

To define formally the morphism $(\alpha \otimes \beta)_M$ consider the following map, which we call $\overline{\alpha}$:

$$(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{U \otimes V \otimes \pi_1} U \otimes V \otimes [V, X] \xrightarrow{U \otimes eval} U \otimes X \xrightarrow{\alpha} N$$

Similarly we define $(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{\overline{\beta}} N$. Then to get $\alpha \otimes \beta$ we pair $\overline{\alpha}$ and $\overline{\beta}$ and use the monoidal structure '$\circ$' of N, as follows:

$$(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{<\overline{\alpha}, \overline{\beta}>} N \times N \xrightarrow{\circ} N$$

Proposition 5 *The construction above induces a bifunctor,*

$$\otimes_M : M_N\mathbf{C} \times M_N\mathbf{C} \to M_N\mathbf{C}$$

covariant in both coordinates, which is a tensor product. The identity I_M is given by $(I \overset{e}{\leftarrow\!\!+} 1)$, where the morphism $I \otimes 1 \cong 1 \xrightarrow{e} N$ just picks up the identity 'e' from the closed poset $(N, \leq, \circ, e, \multimap)$.

Associativity and commutativity of $\otimes_M$ are easy to prove. Note that we are using the tensor product $\otimes$ and the categorical product $\times$ in $\mathbf{C}$, as well as the tensor $\circ$ in N to define the tensor product $\otimes_M$ in $M_N\mathbf{C}$. Note also that $\otimes_M$ is not in general a categorical product, for instance we have no projections, even if $\mathbf{C}$ is a cartesian closed category.

In our previous work with the categories $\mathbf{GC}$, since $\mathbf{C}$ was cartesian closed, it was not clear that only a tensor product was necessary in the first coordinate, whereas a real categorical product was necessary in the second coordinate, to make the definition above work. But as before, this tensor product $\otimes_M$ is designed to make $M_N\mathbf{C}$ monoidal closed, if we consider the following internal-hom.

Definition 9 *Given two objects $A = (U \overset{\alpha}{\leftarrow\!\!+} X)$ and $B = (V \overset{\beta}{\leftarrow\!\!+} Y)$ in $M_N\mathbf{C}$ we define $[A, B]_M$ their internal hom as follows:*

$$[A, B]_M = ([U, V] \times [Y, X] \xleftarrow{(\alpha \multimap \beta)_M} U \otimes Y)$$

The morphism "$(\alpha \multimap \beta)_M$" intuitively says $(\alpha \multimap \beta)_M(\langle f, F \rangle, u \otimes y) = \alpha(u \otimes Fy) \multimap \beta(fu \otimes y)$, where $\multimap$ is the 'internal-hom' in N.

The formal definition of the morphism $(\alpha \multimap \beta)_M$ is similar to the definition of $\otimes_M$ in $M_N\mathbf{C}$. First consider maps $\overline{\alpha}$ and $\overline{\beta}$:

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\pi_1 \otimes U \otimes Y} [U, V] \otimes U \otimes Y \xrightarrow{eval \otimes Y} V \otimes Y \xrightarrow{\beta} N$$

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\pi_2 \otimes U \otimes Y} [Y, X] \otimes U \otimes Y \xrightarrow{U \otimes eval} U \otimes X \xrightarrow{\alpha} N$$

Then to obtain $(\alpha \multimap \beta)_M$ we pair $\overline{\alpha}$ and $\overline{\beta}$ and compose the result with $\multimap$, considered as a map from $N \times N$ to N:

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{<\overline{\alpha}, \overline{\beta}>} N \times N \xrightarrow{\multimap} N$$

As an illustration of how the structure of the closed poset N relates nicely to the categorical structure of $\mathbf{C}$, note that if we consider the internal hom $[A, A]_M$ that is the object

$$([U, U] \times [X, X] \overset{\alpha \multimap \alpha}{\leftarrow\!\!+} U \otimes X)$$

there is always a morphism from I_M to it,

$$\begin{tikzcd}
I \arrow[d] & 1 \arrow[l, "e"'] \\
{[U, U] \times [X, X]} & U \otimes X \arrow[l, "\alpha \multimap \alpha"'] \arrow[u]
\end{tikzcd}$$

as $\mathbf{C}$ is symmetric monoidal closed with products and $e \leq \alpha(u \otimes x) \multimap \alpha(u \otimes x)$.

Proposition 6 *The construction above induces a bifunctor $[-,-]_M$, contravariant in its first coordinate and covariant in its second coordinate.*

Having defined both a tensor product $\otimes_M$ and an internal hom $[-,-]_M$, we want to prove that they provide $M_N\mathbf{C}$ with a symmetric monoidal closed structure.

Theorem 2 *The category $M_N\mathbf{C}$ is a symmetric monoidal closed category.*

The proof is very simple, to verify the natural isomorphism:

$$Hom_{M_N\mathbf{C}}(A \otimes_M B, C) \cong Hom_{M_N\mathbf{C}}(A, [B, C]_M)$$

we look at the diagrams

$$\begin{tikzcd}
U \otimes V \arrow[d, "f"'] & {[V,X] \times [U,Y]} \arrow[l, "\alpha \otimes \beta"'] \\
W & Z \arrow[l, "\gamma"'] \arrow[u, "(f_1, f_2)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, "(f, f_2)"] & X \arrow[l, "\alpha"'] \\
{[V,W] \times [Z,Y]} & V \otimes Z \arrow[l, "\beta \multimap \gamma"'] \arrow[u, "f_1"']
\end{tikzcd}$$

If the morphism $(f, \langle f_1, f_2 \rangle)$ is in $Hom\ (A \otimes B, C)$, then given $u \otimes v$ in $U \otimes V$ and z in Z, we know $(\alpha \otimes \beta)_M(u \otimes v, \langle f_1 z, f_2 z \rangle) \leq \gamma(f(u \otimes v), z)$.

That means, by definition of tensor, that $\alpha(u \otimes f_1 zv) \circ \beta(v \otimes f_2 zu) \leq \gamma(f(u \otimes v) \otimes z)$. But as N is a closed poset,

$$\alpha(u \otimes f_1 zv) \circ \beta(v \otimes f_2 zu) \leq \gamma(f(u \otimes v) \otimes z) \iff \alpha(u \otimes f_1 zv) \leq \beta(v \otimes f_2 zv) \multimap \gamma(f(u \otimes v) \otimes z)$$

Now to show that $(\langle f, f_2 \rangle, f_1)$ is in $Hom(A, [B, C]_M)$ we have to show

$$\alpha(u \otimes f_1(v, z)) \leq (\beta \multimap \gamma)(\langle fu, f_2 u \rangle, v \otimes z)$$

But $(\beta \multimap \gamma)(\langle fu, f_2 u \rangle, v \otimes z) = \beta(v \otimes f_2 uz) \multimap \gamma(fuv \otimes z)$ thus we know exactly what we need to show, if transposing is allowed. □

This proof is basically the same as the one in section 2 for Sets, which shows that we came up with the right definitions for the generalisation proposed.

3.1 Second Comparison

In this section we want compare the symmetric monoidal closed structures of $A_N\mathbf{C}$ and $M_N\mathbf{C}$, but it is slightly easier to compare $A_N\mathbf{Sets}$ and $M_N\mathbf{Sets}$. As we noted before the two categories have the same objects, eg

$$A = U \times X \xrightarrow{\alpha} N \text{ and } B = V \times Y \xrightarrow{\beta} N$$

and for morphisms

$$A_N\mathbf{Sets}(A, B) \subseteq M_N\mathbf{Sets}(A, B)$$

as any morphism in $A_N\mathbf{Sets}$ satisfies not only $\alpha(u, Fy) \leq \beta(fu, y)$ but also the converse. The internal-hom and the tensor product are very similar in shape, but different enough. To make this comparison meaningful we recall the structure of $A_N\mathbf{Sets}$ albeit in a concise way. More details can be found in [Barr79], [Barr91], [L88], [LS'91].

The definition of the internal-hom $[A, B]_A$ in $A_N\mathbf{Sets}$, for objects A and B is given by the object

$$(\mathcal{L}_1(A, B) \xleftarrow{(\alpha \multimap \beta)_A} U \times Y)$$

to use Lafont's notation. Chu writes $\mathcal{L}_1(A, B)$ as $\mathbf{V}(A, B)$. The object $\mathcal{L}_1(A, B)$ is a subset of the internal hom $[U, V] \times [Y, X]$, which as **Sets** is cartesian closed, can be written as $V^U \times X^Y$. The subset $\mathcal{L}_1(A, B)$ is defined by the following pullback:

$$\begin{tikzcd} \mathcal{L}_1(A,B) \arrow[r] \arrow[d] & V^U \arrow[d, "\beta^U"] \\ X^Y \arrow[r, "\alpha^Y"'] & N^{U\times Y} \end{tikzcd}$$

which intuitively means that

$$\mathcal{L}_1(A, B) = \{\langle\phi_1, \phi_2\rangle \mid \phi_1: U \to V, \phi_2: Y \to X \text{ and } \alpha(u, \phi_2 y) = \beta(\phi_1 u, y)\}$$

The diagram above is a pullback so, in particular, it commutes. To define the morphism $(\alpha \multimap \beta)_A$ we want a map

$$(\alpha \multimap \beta)_M : \mathcal{L}_1(A, B) \times U \times Y \to N$$

so nothing more natural than taking the transpose of either of the two composition maps in the square above.

The definition of tensor product in A_N**Sets** is similar, for two objects A and B, $A \otimes_A B$ is given by the object

$$(U \times V \xleftarrow{(\alpha\otimes\beta)_A} \mathcal{L}_2(A, B))$$

where $\mathcal{L}_2(A, B)$, in Lafont's notation is a subset of the internal hom $[V, X] \times [U, Y]$ given by the following pullback:

$$\begin{tikzcd} \mathcal{L}_2(A,B) \arrow[r] \arrow[d] & X^V \arrow[d, "\alpha^V"] \\ Y^U \arrow[r, "\beta^U"'] & N^{U\times V} \end{tikzcd}$$

which intuitively means that

$$\mathcal{L}_2(A, B) = \{\langle\phi_1, \phi_2\rangle \mid \phi_1: V \to X, \phi_2: U \to Y \text{ and } \alpha(u, \phi_1 v) = \beta(v, \phi_2 u)\}$$

That is equivalent to saying that $\mathcal{L}_2(A, B)$ can be 'represented' by maps $\phi: U \times V \to N, \phi_1: U \to Y, \phi_2: V \to X$ such that

$$\alpha(u, \phi_2 v) = \beta(v, \phi_1 u) = \phi(u, v)$$

But note that the map $U \times V \to N$ is redundant, as it can be defined in terms of ϕ_1 and ϕ_2.

As before, to define the tensor we want a map

$$\alpha \otimes \beta: U \times V \times \mathcal{L}_2(A, B) \to N$$

so we take the transpose of either of the two composition maps in the pullback square.

Chu does not explicitly describe the tensor product, as it can be defined in terms of negation, as $A \otimes_A B = [A, B^\perp]^\perp$

Since $A_N\mathbf{C}$ is a subcategory of $M_N\mathbf{C}$, the two symmetric monoidal closed structures can be compared within $M_N\mathbf{C}$.

Proposition 7 *The tensor products $(\alpha \otimes \beta)_M$ and $(\alpha \otimes \beta)_A$ can be related by the following diagram:*

$$
\begin{tikzcd}
U \times V \arrow[d, "1"'] & X^V \times Y^U \arrow[l, "(\alpha \otimes \beta)_M"'] \\
U \times V & \mathcal{L}_2(A, B) \arrow[l, "(\alpha \otimes \beta)_A"'] \arrow[u, "i"']
\end{tikzcd}
$$

Similarly the internal homs can be related via the diagram

$$
\begin{tikzcd}
\mathcal{L}_1(A, B) \arrow[d, "i"'] & U \times Y \arrow[l, "(\alpha \multimap \beta)_A"'] \\
V^U \times X^Y & U \times Y \arrow[l, "(\alpha \multimap \beta)_M"'] \arrow[u, "1"']
\end{tikzcd}
$$

In particular, the identities for the tensors can also be compared

$$
\begin{tikzcd}
1 \arrow[d, "1"'] & 1 \arrow[l, "i"'] \\
1 & N \arrow[l, "id"'] \arrow[u, "!"']
\end{tikzcd}
$$

One very nice thing about the category A_NSets or $GAME_K$ is its relationship to Linear Algebra, cf. [LS'91]. Seely remarks in [See] that when Chu was writing about symmetric monoidal closed categories, Linear Logic had not been invented by Girard. Hence there is nothing about additives - nor about a '!' comonad - in Chu's original construction. Also, when following Girard's and Hyland's suggestions in Boulder 87, I wrote about the categories **GC** as models of Linear Logic, I knew nothing about Chu's construction. But additives were very easy to construct in **GC**, as they are in M_N**C**.

3.2 Additive Structure in M_NC

If **C** has finite coproducts - as well as being symmetric monoidal closed with products - and N is a closed poset as before, products and coproducts in M_N**C** are very easy to define using their counterparts in **C**. The method is the same used for **GC** and subsequently for $GAME_K$ and A_NSets.

Definition 10 *Given two objects $(U \overset{\alpha}{\leftarrow} X)$ and $(V \overset{\beta}{\leftarrow} Y)$ in M_N**C** we define their categorical product as follows:*

$$A \& B = (U \times V \overset{\alpha\&\beta}{\leftarrow} X + Y)$$

The morphism "$\alpha\&\beta$" is given intuitively by $\alpha\&\beta(\langle u, v\rangle, \binom{x,0}{y,1}) = \alpha(u, x) \cdot \beta(v, y)$

But we do no operation to $\alpha(u,x)$ and $\beta(v,y)$, as we either have $(x,0)$ or $(y,1)$, but never both, by definition of the coproduct $X+Y$.

More precisely $\alpha\&\beta$ is given by the morphism

$$(U\times V)\otimes(X+Y)\cong(U\times V)\otimes X+(U\times V)\otimes Y \xrightarrow{`\pi_1+\pi_2'} U\otimes X+V\otimes Y \xrightarrow{\binom{\alpha}{\beta}} N$$

It is easy to check that this operation defines a bifunctor with identity given by $1_M=(1 \dot{\leftrightarrow} 0)$ - the empty multirelation - and that $\alpha\&\beta$ is a categorical product. The projections are projections in the first coordenate and canonical injections in the second coordinate. Similarly we have coproducts.

Definition 11 *Given two objects* $(U \overset{\alpha}{\leftrightarrow} X)$ *and* $(V \overset{\beta}{\leftrightarrow} Y)$ *in* $M_N\mathbf{C}$ *we define their categorical coproduct*

$$A\oplus B=(U+V \overset{\alpha\oplus\beta}{\leftrightarrow} X\times Y)$$

The morphism "$\alpha\oplus\beta$*" is given by* $\alpha\oplus\beta(\binom{u,0}{v,1},\langle x,y\rangle)=\alpha(u,x)\cdot\beta(v,y)$

Again it is easy to check that the bifunctor "$\oplus$" provides categorical coproducts and that 0_M given by $(0 \dot{\leftrightarrow} 1)$ is the initial object.

Proposition 8 *The category* $M_N\mathbf{C}$ *has binary products and coproducts.*

It is clear that the category $M_N\mathbf{C}$ above provides a model for Intuitionistic Linear Logic, as described in the appendix.

Theorem 3 *The category* $M_N\mathbf{C}$ *is a categorical model of Intuitionistic Linear Logic.*

The proof is trivial, as the constants $I_M, 1_M, 0_M$ and bifunctors $\otimes, \multimap, \&, \oplus$ were defined for it. $\square$

Observe that the additive structure of $A_N\mathbf{C}$ is the same as that of $M_N\mathbf{C}$ - or **GC** for that matter.

4 Modalities in $M_N\mathbf{C}$

This section should be considered as 'work in progress', as we really would like to have the results for a category **C** symmetric monoidal closed with products, instead of for a cartesian closed category. But the calculations seem to be correct and the case of **C** cartesian closed is the important one for the Petri Nets applications.

If we assume that **C** is a cartesian closed category - thus a fortiori a symmetric monoidal closed category with products - with free commutative monoids we can provide the linear logic modality '!' for $M_N\mathbf{C}$ as a model of Intuitionistic Linear Logic. Note that, in particular the category **Sets** satisfies all these conditions.

The general idea - analogous once more to the previous work on **CG** - is to define comonads T and S in $M_N\mathbf{C}$ and compose them to get another comonad called suggestively '!' in $M_N\mathbf{C}$. But comonads T and S come from monads $(-)^U$, $(-)^*$ and their composite $(-)^{*U}$, in **C**. Thus we have subsections for T, S and '!', as well as one subsection on the logical properties of the comonads.

Recall that for **C** a cartesian closed category $M_N\mathbf{C}$ simplifies slightly as it has as objects maps $U\times X \xrightarrow{\alpha} N$ and as morphisms pairs of maps (f,F) in **C** $f\colon U\to V$ and $F\colon Y\to X$, such that in the following diagram

$$
\begin{tikzcd}
U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] & U \times X \arrow[d, "\alpha"] \\
V \times Y \arrow[r, "\beta"'] & N
\end{tikzcd}
$$

we have $\alpha \circ (U \times F) \leq (f \times Y) \circ \beta$. Also we recap briefly our constructions of the last section, for **C** cartesian closed:

- the tensor product $A \otimes_M B$ in $M_N\mathbf{C}$ is given by

$$(U \times V \overset{\alpha\otimes\beta}{\leftarrow\!\!\!\!+} X^Y \times Y^X)$$

  with identity $I = (1 \overset{0}{\leftarrow\!\!\!\!+} 1)$;

- the internal hom $[A, B]_M$ is given by

$$(V^U \times X^Y \overset{\alpha\multimap\beta}{\leftarrow\!\!\!\!+} U \times Y);$$

- categorical products $A \& B$ are

$$(U \times V \overset{\alpha\&\beta}{\leftarrow\!\!\!\!+} X + Y)$$

  with identity $I = (1 \leftarrow\!\!\!\!+ 0)$;

- and coproducts $A \oplus B$ are

$$(U + V \overset{\alpha\oplus\beta}{\leftarrow\!\!\!\!+} X \times Y)$$

4.1 The comonad T

The comonad T is as easy to define for $M_N\mathbf{C}$ as it was for **GC**. Recall that any *fixed* object U in a cartesian closed category **C** induces an endofunctor

$$(\ )^U : \mathbf{C} \to \mathbf{C}$$

$$
\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & X^U \arrow[d, "f^U"] \\
Y \arrow[r, mapsto] & Y^U
\end{tikzcd}
$$

This endofunctor has a natural monad structure where the unit of the monad $X \xrightarrow{\eta_1} X^U$ is given by the transpose of the second projection $U \times X \xrightarrow{\pi_2} X$ and the monad multiplication $(X^U)^U \xrightarrow{\mu_1} X^U$ is given by precomposing with the diagonal map $\Delta: U \to U \times U$, thus $(X^U)^U \cong X^{U\times U} \xrightarrow{X^\Delta} X^U$. We summarize that in the definition below.

Definition 12 *For each object U in a cartesian closed category $\mathbf{C}$ we have a monad $((\ )^U, \eta_1, \mu_1)$ in $\mathbf{C}$ given by the natural transformations below:*

$$X \xrightarrow{\eta_1} X^U \qquad\qquad X^{U\times U} \xrightarrow{\mu_1} X^U$$

As they are monads, the endofunctors $(\ )^U$ make the following diagrams commute:

$$\begin{tikzcd}
X^U \arrow[r, "\eta_1 X^U"] \arrow[dr] & X^{U\times U} \arrow[d, "\mu_1"'] & X^U \arrow[l, "(\eta_1)^U"'] \arrow[dl] \\
& X^U &
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{U\times U\times U} \arrow[r] \arrow[d, "\mu_1^U"'] & X^{U\times U} \arrow[d, "\mu_1"] \\
X^{U\times U} \arrow[r, "\mu_1"'] & X^U
\end{tikzcd}$$

One important fact about the monads $(-)^U$ is that they also make the following diagrams commute.

Fact 1 *The following diagrams commute.*

$$\begin{tikzcd}
U\times X \arrow[r, "U\times\eta_1"] \arrow[dr] & U\times X^U \arrow[d, "\langle\pi_1, ev\rangle"] \\
& U\times X
\end{tikzcd}
\qquad
\begin{tikzcd}
U\times X^{U\times U} \arrow[r, "U\times\mu_1"] \arrow[d, "\langle\pi_1, ev\rangle"'] & U\times X^U \arrow[d, "\langle\pi_1, ev\rangle"] \\
U\times X^U \arrow[r, "\langle\pi_1, ev\rangle"'] & U\times X
\end{tikzcd}$$

That is a consequence of the fact that $(\ )^U$ is the monad induced by the adjunction

$$< \Delta_U, \Pi_U, \eta, \varepsilon >: \mathbf{C} \to \mathbf{C}[U]$$

also written as $\Delta_U \dashv \Pi_U$, cf. [LSc'86]. We use the monads $(\ )^U$ in $\mathbf{C}$ to define the comonad T in $M_N\mathbf{C}$ and the fact above is used to show that T has a comonad structure.

Definition 13 *The endofunctor $T: M_N\mathbf{C} \to M_N\mathbf{C}$ takes an object $(U \overset{\alpha}{\nrightarrow} X)$ of $M_N\mathbf{C}$ to the object $(U \overset{T\alpha}{\nrightarrow} X^U)$, where intuitively the object $T\alpha$ is given by $T\alpha(u, f) = \alpha(u, fu)$.*

In other words, the object $T\alpha$ is given by the following composition:

$$U\times X^U \xrightarrow{\langle\pi_1, ev\rangle} U\times X \xrightarrow{\alpha} N$$

If $(f, F): A \to B$ is a morphism in $M_N\mathbf{C}$, then $T(f, F)$ is given by

$$\begin{tikzcd}
U \arrow[d, "f"'] & X^U \arrow[l, "T\alpha"'] \\
V & Y^V \arrow[l, "T\beta"] \arrow[u, "F\circ(\ )\circ f"']
\end{tikzcd}$$

To show that T is an endofunctor in $M_N\mathbf{C}$ we have to check the following diagram,

$$\begin{tikzcd}
U \times Y^V \arrow[r, "U \times Y^f"] \arrow[dd, "f \times Y^V"'] & U \times Y^U \arrow[r, "U \times F^U"] \arrow[d, "\langle \pi_1, ev \rangle"'] \arrow[dr, phantom, "="] & U \times X^U \arrow[d, "\langle \pi_1, ev \rangle"] \\
 & U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] \arrow[dr, phantom, "\geq"] & U \times X \arrow[d, "\alpha"] \\
V \times Y^V \arrow[r, "\langle \pi_1, ev \rangle"'] & V \times Y \arrow[r, "\beta"'] & N
\end{tikzcd}$$

The functor T has a natural comonad structure inherited from the monoidal structure of the functors $(-)^U$ for U in $\mathbf{C}$. Thus we have natural transformations $\epsilon_1 : TA \to A$ and $\delta_1 : TA \to T^2A$ in $M_N\mathbf{C}$ given by,

$$\begin{tikzcd}
U \arrow[d, "1"'] & X^U \arrow[l, "T\alpha"'] \\
U & X \arrow[l, "\alpha"] \arrow[u, "\eta_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, "1"'] & X^U \arrow[l, "T\alpha"'] \\
U & X^{U \times U} \arrow[l, "T^2\alpha"] \arrow[u, "\mu_1"']
\end{tikzcd}$$

To show that these are morphisms in $M_N\mathbf{C}$ we note that the following diagrams commute:

$$\begin{tikzcd}
U \times X \arrow[r, "U \times \eta_1"] \arrow[d, "1"'] & U \times X^U \arrow[d, "T\alpha"] \\
U \times X \arrow[r, "\alpha"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times X^{U \times U} \arrow[r, "U \times \mu_1"] \arrow[d, "1"'] & U \times X^U \arrow[d, "T\alpha"] \\
U \times X^{U \times U} \arrow[r, "T^2\alpha"'] & N
\end{tikzcd}$$

Commutativity here is a consequence of the commutativity of the diagrams in fact 1.

4.2 The comonad S

Now to define a comonad S we assume free commutative monoids in $\mathbf{C}$, analogous to what we did for the categories $\mathbf{GC}$.

Suppose $\mathbf{C}$ has *strong commutative free monoids*. By that we mean that there exists a functor $F : \mathbf{C} \to \mathbf{Mon_C C}$, which is left-adjoint to the forgetful functor $U : \mathbf{Mon_C C} \to \mathbf{C}$ and the monad induced by this adjunction is a *strong* one. In more detail, recall that:

Fact 2 *The category* $\mathbf{Mon_C C}$ *consists of (commutative) monoid objects in* $\mathbf{C}$*. That is objects in* $\mathbf{Mon_C C}$ *are triples* (Y, η_Y, μ_Y) *where* $\eta_Y : 1 \to Y$ *and* $\mu_Y : Y \times Y \to Y$ *are morphisms in* $\mathbf{C}$ *such* *that the following diagrams commute:*

$$
\begin{tikzcd}
Y \times 1 \arrow[r, "\eta"] \arrow[dr] & Y \times Y \arrow[d, "\mu"'] & 1 \times Y \arrow[l, "(\eta)"'] \arrow[dl] \\
 & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times Y \times Y \arrow[r] \arrow[d, "\mu"'] & Y \times Y \arrow[d] \\
Y \times Y \arrow[r] & Y
\end{tikzcd}
$$

Morphisms in $\mathbf{Mon_C C}$ *are morphisms in* $\mathbf{C}$, *which preserve the monoidal structure.*

There is an adjunction $< F, U, \eta, \varepsilon >: \mathbf{C} - \mathbf{Mon_C C}$, also writen as $F \dashv U$, which says that a map in $\mathbf{C}$ corresponds by a natural isomorphism, to a monoid homomorphism $\overline{f}$ in $\mathbf{Mon_C C}$, as follows:

$$
\frac{X \xrightarrow{f} U(Y, \eta_Y, \mu_Y)}{(X^*, \eta_{X^*}, \mu_{X^*}) \xrightarrow{\overline{f}} (Y, \eta_Y, \mu_Y)}
$$

We write $(\ )^*$ for the composite functor $U \cdot F: \mathbf{C} - \mathbf{C}$. Thus we have an endofunctor ${}^*: \mathbf{C} \to \mathbf{C}$ given by,

$$
\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & X^* \arrow[d, "f^*"] \\
Y \arrow[r, mapsto] & Y^*
\end{tikzcd}
$$

Intuitively, X^* consists of commutative finite sequences of elements of X, hence we write an element of X^* as $\overline{x}$, meaning $\overline{x} = \langle x_1, x_2, \ldots, x_k \rangle$ and f^* is just f in each coordinate. We summarize in the following definition.

Definition 14 *We denote by* $((\ )^*, \eta_2, \mu_2)$ *the (strong) monad in* $\mathbf{C}$ *corresponding to the adjunction* $F \dashv U$, *which is given by the endofunctor* $(\ )^* = U \cdot F$ *and the natural transformations*

$$
X \xrightarrow{\eta_2} X^* \qquad X^{**} \xrightarrow{\mu_2} X^*
$$

satisfying:

$$
\begin{tikzcd}
X^* \arrow[r, "\eta_2"] \arrow[dr] & X^{**} \arrow[d, "\mu_2"'] & X^* \arrow[l, "(\eta_2)^*"'] \arrow[dl] \\
 & X^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
X^{***} \arrow[r] \arrow[d, "\mu_2^*"'] & X^{**} \arrow[d, "\mu_2"] \\
X^{**} \arrow[r, "\mu_2"'] & X^*
\end{tikzcd}
$$

The unit of the adjunction $F \dashv U$, the natural transformation $\eta_2 \colon \mathbf{C} \dot{\rightarrow} \mathbf{C}$ takes any object X of $\mathbf{C}$ to the carrier of the free commutative monoid X^*. Intuitively η_2 makes a singleton sequence of the element x of X and μ_2 transforms a sequence of sequences into a sequence.

Note that each free commutative monoid (X^*, η_*, μ_*) comes equipped with maps in $\mathbf{C}$, $1 \xrightarrow{\eta_*} X^*$ and $X^* \times X^* \xrightarrow{\mu_*} X^*$ where η_* intuitively picks the empty sequence $\langle\ \rangle$ of elements of X and μ_* concatenates - commutatively - sequences of elements of X $\overline{x}_1$ and $\overline{x}_2$. The following diagrams commute:

$$\begin{tikzcd}
1 \times X^* \arrow[r] \arrow[dr] & X^* \times X^* \arrow[d, "\mu_*"] & X^* \times 1 \arrow[l] \arrow[dl] \\
& X^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
X^* \times X^* \times X^* \arrow[r] \arrow[d, "\mu_* \times X^*"'] & X^* \times X^* \arrow[d, "\mu_*"] \\
X^* \times X^* \arrow[r, "\mu_*"'] & X^*
\end{tikzcd}$$

The co-unit of the adjunction $\epsilon \colon \mathbf{Mon_C C} \dot{\rightarrow} \mathbf{Mon_C C}$ takes any free commutative monoid $(X^*, \eta*, \mu*)$ arising from an arbitrary commutative monoid (X, η, μ) to itself. Thus

$$\epsilon \colon FU(M, \eta, \mu) = (M^*, \eta*, \mu*) \rightarrow (M, \eta, \mu)$$

where the morphism ϵ corresponds to 'iteration' of the original multiplication μ.

If the category $\mathbf{C}$ has a 0 object, then we have $0^* \cong 1$. In particular, if we are thinking of $\mathbf{C}$ as Sets and N the set of natural numbers with its additive monoidal structure $(\mathbf{N}, 0, +)$, as in section 2, we have $\epsilon(N^*, \eta_2, \mu_2) = (N, 0, +)$. Intuitively that says $\epsilon(\langle\ \rangle) = 0$ and $\epsilon(\langle n_1, n_2, \ldots, n_k \rangle) = n_1 + n_2 + \ldots + n_k$, which implies $\epsilon(\langle n \rangle) = n$. More in general, if N is the closed poset $(N, \leq, \circ, e, \multimap)$ we have $\epsilon(N^*, \eta_*, \mu_*) = (N, e, \circ)$ and

$$\epsilon(\langle n_1, n_2, \ldots, n_k \rangle) = n_1 \circ n_2 \circ \ldots \circ n_k \text{ and } \epsilon(\langle\ \rangle) = e.$$

In this stronger version the monad $((\ )^*, \eta, \mu)$ is a *strong monad*, so there are morphisms

$$Y^X \xrightarrow{st} Y^{*X^*}$$

Because we are considering free *commutative* monoids in $\mathbf{C}$ we have

$$(X + Y)^* \cong X^* \times Y^*$$

We also have a natural transformation $m \colon U^* \times X^* \rightarrow (U \times X)^*$. Intuitively m takes a sequence of u's and a sequence of x's to a sequence where each u is followed by a single x as follows:

$$m(\overline{u}, \overline{x}) = \langle u_1\overline{x}, \ldots, u_k\overline{x} \rangle = \langle u_1x_1, \ldots, u_1x_m, \ldots u_kx_1, \ldots u_kx_m \rangle$$

Using the natural transformations m we can define the endofunctor S below.

Definition 15 *The endofunctor S:$M_N\mathbf{C} \rightarrow M_N\mathbf{C}$ takes an object $(U \overset{\alpha}{\nrightarrow} X)$ of $M_N\mathbf{C}$ to the object $(U \overset{S\alpha}{\nrightarrow} X^*)$, where, as intuitively $\overline{x}$ is $\langle x_1, x_2, \ldots, x_n \rangle$, $S\alpha(u, \overline{x})$ means $\alpha(u, x_1)$ and $\alpha(u, x_2)$ and $\ldots$ and $\alpha(u, x_n)$.*

The object $S\alpha$ of $M_N\mathbf{C}$ is defined by the long morphism

$$U \times X^* \xrightarrow{\eta_2 \times X^*} U^* \times X^* \xrightarrow{m} (U \times X)^* \xrightarrow{\alpha^*} N^* \xrightarrow{\epsilon} N$$

If $(f, F): A \to B$ is a morphism in $M_N\mathbf{C}$, $S(f, F)$ is given by (f, F^*) as follows

$$
\begin{tikzcd}
U \arrow[r, dash, "S\alpha"] \arrow[d, "f"'] & X^* \\
V & Y^* \arrow[l, "S\beta"] \arrow[u, "F^*"']
\end{tikzcd}
$$

As an illustration of the conciseness of the notation used, recall that the small diagram above corresponds to the big one below:

$$
\begin{tikzcd}
U \times Y^* \arrow[rr] \arrow[d] & & U \times Y^* \arrow[r, "U \times F^*"] \arrow[d, "\eta_2 \times Y^*"] & U \times X^* \arrow[d, "\eta_2 \times X^*"] & \\
\cdot \arrow[d] & & U^* \times Y^* \arrow[r, "U^* \times F^*"] \arrow[d, "m"] & U^* \times X^* \arrow[d, "m"] & \\
U \times Y^* \arrow[r, "\eta_2 \times Y^*"] \arrow[d, "f \times Y^*"'] & U^* \times Y^* \arrow[r, "m"] \arrow[d, "f^* \times Y^*"] & (U \times Y)^* \arrow[r, "(U \times F)^*"] \arrow[d, "(f \times Y)^*"] & (U \times X)^* \arrow[d, "\alpha^*"] & \\
V \times Y^* \arrow[r, "\eta_2 \times Y^*"'] & V^* \times Y^* \arrow[r, "m"'] & (V \times Y)^* \arrow[r, "\beta^*"'] & N^* \arrow[dr, "\epsilon"] & \\
& & & & N
\end{tikzcd}
$$

To show that S is an endofunctor, remember that for the square most down to the right, we use that the functor $(\ )^*$ preserves the order on morphisms in $\mathbf{C}$, for the other squares we have equality.

The endofunctor $S\alpha$ has a natural comonad structure given by the monad structure of $(\ )^*$ in $\mathbf{C}$. Thus we have morphisms $\epsilon_2: SA \to A$ and $\delta_2: SA \to S^2A$ in $M_N\mathbf{C}$ given by

$$
\begin{tikzcd}
U & X^* \arrow[l, "S\alpha"'] \\
U \arrow[u, leftarrow, "1"] & X \arrow[l, "\alpha"] \arrow[u, "\eta_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U & X^* \arrow[l, "S\alpha"'] \\
U \arrow[u, leftarrow, "1"] & X^{**} \arrow[l, "S^2\alpha"] \arrow[u, "\mu_2"']
\end{tikzcd}
$$

To show that the above are morphisms ϵ_2 and δ_2 in $M_N\mathbf{C}$ we note that the diagrams below commute, which is a consequence of the following fact.

$$
\begin{tikzcd}
U \times X \arrow[r, "U \times \eta_2"] \arrow[d, "1"'] & U \times X^* \arrow[d, "S\alpha"] \\
U \times X \arrow[r, "\alpha"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times X^{**} \arrow[r, "U \times \mu_2"] \arrow[d, "1"'] & U \times X^* \arrow[d, "S\alpha"] \\
U \times X^{**} \arrow[r, "S^2\alpha"'] & N
\end{tikzcd}
$$

Fact 3 *The following diagrams commute:*

$$
\begin{tikzcd}
U \times X \arrow[r, "U \times \eta_2"] \arrow[d, "\alpha"'] & U \times X^* \arrow[r, "\eta_2 \times X^*"] \arrow[dl, "S\alpha"'] & U^* \times X^* \arrow[d, "m"] \\
N & N^* \arrow[l, "\epsilon"'] & (U \times X)^* \arrow[l, "\alpha^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
U^* \times X^* \arrow[r, "m"] & (U \times X)^* \arrow[d, "\alpha^*"] \\
U \times X^* \arrow[u, "\eta \times X^*"] \arrow[dr, "S\alpha"] & N^* \arrow[d] \\
U \times X^{**} \arrow[u, "U \times \mu"] \arrow[r, "S^2\alpha"] \arrow[d, "\eta \times X^{**}"'] & N \\
U^* \times X^{**} \arrow[d, "m"'] & \\
(U \times X^*)^* \arrow[r, "(S\alpha)^*"] & N^* \arrow[uu]
\end{tikzcd}
$$

Note that the first diagram commutes because the transformation m applied to singletons is a singleton, $m(\langle u \rangle, \langle x \rangle) = \langle u\, x \rangle$; α^* applied to a singleton is α and $\epsilon(\langle n \rangle) = n$. The second diagram says we can transform a sequence of sequences before applying the endofunctor S a second time.

4.3 The comonad '!'

Now we want to compose the two comonads T and S above, that is we want to define ! as $S \circ T$. To give an intuitive definition is easy:

Definition 16 *The endofunctor* $!: M_N\mathbf{C} \to M_N\mathbf{C}$ *takes an object* $(U \stackrel{\alpha}{\to} X)$ *of* $M_N\mathbf{C}$ *to the object* $(U \stackrel{!\alpha}{\to} X^{*U})$, *where intuitively if* $\phi: U \to X^*$ *and* $\phi u = \langle x_1, x_2, \ldots, x_n \rangle$ *then* $!\alpha(u,\phi)$ *is given by* $\alpha(u, x_1)$ *and* $\alpha(u, x_2)$ *and ... and* $\alpha(u, x_n)$.

But to define $!\alpha$ formally is a long process. First we define a composite monad $(\ )^{*U}$ in $\mathbf{C}$.
Note that there is an endofunctor in $\mathbf{C}$ given by the composition of the monads $(\ )^U$ and $(\ )^*$. Thus $(\ )^{*U}: \mathbf{C} \to \mathbf{C}$ takes

$$\begin{array}{ccc} X & \mapsto & (X^*)^U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f^*)^U} \\ Y & \mapsto & (Y^*)^U \end{array}$$

The endofunctor $(\ )^{*U}$ has a natural monad structure in $\mathbf{C}$, its unit $X \xrightarrow{\eta_3} X^{*U}$ is given by the composition of the units η_2 and η_1 as follows,

$$X \xrightarrow{\eta_2} X^* \xrightarrow{\eta_1} (X^*)^U$$

To give a multiplication $\mu_3: X^{*U*U} \to X^{*U}$ we first note:

- We have the following series of natural transformations in $\mathbf{C}$:

$$\frac{\dfrac{\dfrac{U \times X^U \xrightarrow{\eta_2} (U \times X^U)^*}{X^U \longrightarrow (U \times X^U)^{*U}}}{(X^U)^* \longrightarrow (U \times X^U)^{*U}}}{U \times (X^U)^* \xrightarrow{a} (U \times X^U)^*}$$

  From the first line to the second, we just take the exponential transpose. From the second to the third, we use the fact that if Y has a monoid structure, the same happens to Y^U - Y is $(U \times X^U)^*$ in this case - and the free monoids adjunction. The last step is just exponential transposition again.

- There is a natural transformation in $\mathbf{C}$ given by

$$\lambda_X: (X^U)^* \to (X^*)^U$$

Definition 17 *To obtain the natural transformation* λ *it is enough to have the auxiliary map* $a: U \times (X^U)^* \to (U \times X^U)^*$ *above, as we could compose it with* $ev^*: (U \times X^U)^* \to X^*$ *and take the transpose.*

$$\frac{U \times (X^U)^* \xrightarrow{a} (U \times X^U)^* \xrightarrow{ev^*} X^*}{(X^U)^* \xrightarrow{\lambda_X} X^{*U}}$$

This λ is a distributive law of monads [Beck]. Thus we can use Beck's results and

- Finally we say that μ_3 is given by the transpose of the long composition:

$$\frac{X^{\bullet U^{\bullet U}} \times U \xrightarrow{\langle ev, \pi_2 \rangle} X^{\bullet U^{\bullet}} \times U \xrightarrow{\lambda \times U} (X^{\bullet\bullet})^U \times U \xrightarrow{ev} X^{\bullet\bullet} \xrightarrow{\mu_2} X^{\bullet}}{X^{\bullet U^{\bullet U}} \xrightarrow{\mu_3} X^{\bullet U}}$$

Note that to define the multiplication μ_3 we use μ_2 and the distributive law λ, as well as evaluation on U twice, instead of μ_1. Note as well that as an endofunctor the composition $(\ )^{U^{\bullet}}$ also makes sense, but it has no natural monad structure. We summarize the discussion above in the definition,

Definition 18 *The endofunctor $(\ )^{\bullet U}$ has a natural monad structure in* $\mathbf{C}$*, with unit and multiplication given by the natural transformations*

$$X \xrightarrow{\eta_3} X^{\bullet U} \qquad\qquad X^{\bullet U^{\bullet U}} \xrightarrow{\mu_3} X^{\bullet U}$$

We need another fact, a similar result was proved in the the work on the categories $\mathbf{GC}$:

Fact 4 *The distributive law λ in* $\mathbf{C}$ *induces a distributive law of comonads Λ in $M_N\mathbf{C}$, given by $\Lambda: TSA \to STA$:*

$$\begin{tikzcd}
U & X^{\bullet U} \arrow[l, "TS\alpha"'] \\
U \arrow[u, leftarrow, "1"] & X^{U^{\bullet}} \arrow[l, "ST\alpha"] \arrow[u, "\lambda"']
\end{tikzcd}$$

To check that Λ is a morphism in $M_N\mathbf{C}$, we check the diagram

$$\begin{tikzcd}
U \times (X^U)^{\bullet} \arrow[r, "U \times \lambda"] \arrow[d, "1"'] & U \times X^{\bullet U} \arrow[d, "TS\alpha"] \\
U \times (X^U)^{\bullet} \arrow[r, "ST\alpha"'] & N
\end{tikzcd}$$

The composition monad $(-^{\bullet})^U$ above induces an endofunctor in $M_N\mathbf{C}$, which is the composition of S and T. We call this endofunctor '!', its intuitive definition was given before. More formally,

Definition 19 *The endofunctor ! in $M_N\mathbf{C}$ acts on objects as*

$$!(U \overset{\alpha}{\multimap} X) = (U \overset{!\alpha}{\multimap} (X^{\bullet})^U)$$

where the morphism $!\alpha$ is given by composition

$$U \times (X^{\bullet})^U \xrightarrow{\langle \pi, ev \rangle} U \times X^{\bullet} \xrightarrow{S\alpha} N$$

If $(f, F): A \to B$ is a morphism in $M_N\mathbf{C}$, then $!(f, F)$ is given by

$$\begin{tikzcd}
U \arrow[d, "f"'] & X^{\bullet U} \arrow[l, "!\alpha"'] \\
V & Y^{\bullet V} \arrow[l, "!\beta"] \arrow[u, "F^{\bullet} \cdot (\ ) \cdot f"']
\end{tikzcd}$$

To show that ! is really an endofunctor we have to compose the squares we had before for T and S.

$$\begin{tikzcd}
U \times Y^{\bullet V} \arrow[r, "U \times Y^{f}"] \arrow[dd, "f \times Y^{V}"'] & U \times Y^{\bullet U} \arrow[r, "U \times F^{\bullet U}"] \arrow[d, "\langle \pi_1, ev \rangle"'] & U \times X^{\bullet U} \arrow[d, "\langle \pi_1, ev \rangle"] \\
 & U \times Y^{\bullet} \arrow[r, "U \times F^{\bullet}"] \arrow[d, "f \times Y^{\bullet}"'] & U \times X^{\bullet} \arrow[d, "S\alpha"] \\
V \times Y^{\bullet V} \arrow[r, "\langle \pi_1, ev \rangle"'] & V \times Y^{\bullet} \arrow[r, "S\beta"'] & N
\end{tikzcd}$$

The endofunctor ! in $M_N\mathbf{C}$ has a natural comonad structure given by the monad structure of $(\ )^{\bullet U}$ in $\mathbf{C}$. Thus $\epsilon_{!}: !A \to A$ and $\delta_{!}: !A \to !!A$ are given by

$$\begin{tikzcd}
U \arrow[d, "1"'] & X^{\bullet U} \arrow[l, "!\alpha"'] \\
U & X \arrow[l, "\alpha"] \arrow[u, "\eta_3"']
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[d, "1"'] & X^{\bullet U} \arrow[l, "!\alpha"'] \\
U & X^{\bullet U \bullet U} \arrow[l, "!!\alpha"'] \arrow[u, "\mu_3"']
\end{tikzcd}$$

To show that these maps are maps in $M_N\mathbf{C}$ is just the composition of the diagrams we have shown to commute for T and S. Thus

$$\begin{tikzcd}
U \times X \arrow[r, "U \times \eta_3"] \arrow[d, "1"'] & U \times X^{\bullet U} \arrow[d, "TS\alpha"] \\
U \times X \arrow[r, "\alpha"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times X^{\bullet U \bullet U} \arrow[r, "U \times \mu_3"] \arrow[d] & U \times X^{\bullet U} \arrow[d] \\
U \times X^{\bullet U \bullet U} \arrow[r] & N
\end{tikzcd}$$

After all the work above to define the comonad '!' we must show that it works. We do that in the next section, but before heading for the logic we need a last categorical proposition.

Proposition 9 *The comonad '!' in $M_N\mathbf{C}$ defined above satisfies*

$$!(A \& B) \cong !A \otimes !B \qquad \textit{and} \qquad !1 \cong I$$

Proof: By definition of '!' we have:

$$!A = (U \overset{!\alpha}{\leftarrow} X^{\cdot U}) \qquad !B = (V \overset{!\beta}{\leftarrow} Y^{\cdot V}) \qquad !1 = (1 \overset{!}{\leftarrow} 0^{\cdot 1})$$

By definition of the product $A\&B$,

$$!(A\&B) = !(U \times V \overset{\alpha\&\beta}{\leftarrow} X + Y) = (U \times V \overset{!(\alpha\&\beta)}{\leftarrow} (X+Y)^{\cdot U \times V})$$

Taking the tensor product we have

$$!A \otimes !B = (U \times V \overset{!\alpha \otimes !\beta}{\leftarrow} X^{\cdot U \times V} \times Y^{\cdot V \times U})$$

Thus to show the isomorphisms in $M_N\mathbf{C}$ we have to show the following isomorphisms in $\mathbf{C}$,

$$(X+Y)^{\cdot U \times V} \cong X^{\cdot U \times V} \times Y^{\cdot V \times U} \qquad 0^{\cdot} \cong 1$$

and that these isomorphisms induce isomorphisms in $M_N\mathbf{C}$. The isomorphisms in $\mathbf{C}$ are clear from that fact that $(X+Y)^{\cdot} \cong X^{\cdot} \times Y^{\cdot}$, which implies that

$$(X+Y)^{\cdot U \times V} \cong X^{\cdot U \times V} \times Y^{\cdot U \times V}$$

Actually that is the reason why we are taking commutative monoids in $\mathbf{C}$. □

4.4 Logical Properties of '!'

Now to show the logical properties of the comonad '!' we first recall the rules for the modality *of course!* in Linear Logic.

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (\textit{dereliction}) \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\textit{weakening})$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\textit{contraction}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!)$$

Our next theorem show that the comonad '!' defined in the last section really works. The details are very similar to our previous work as well as to Seely's work [See'87], to which we refer the reader. The basic idea is to show that the rules are sound by showing that, if there is a morphism in the category $M_N\mathbf{C}$ between the objects which are the translation of the antecedent, then there is a morphism between the objects which translate the sucedent of each rule.

Theorem 4 *The comonad '!' in $M_N\mathbf{C}$ satisfies the rules for the modality '!' in Linear Logic.*

It is clear that by virtue of being a comonad '!' satisfies the rule (*dereliction*). To wit, if there is always a morphism $!A \xrightarrow{\eta} A$, whenever we have a map $G \otimes A \xrightarrow{f} B$ we can compose it with $G \otimes !A \xrightarrow{G\otimes\eta} G \otimes A$ to get $G \otimes !A \longrightarrow B$, which shows that the rule (*dereliction*) is sound.

To show soundness of the rule (!) we need more. If there is always a map $!A \xrightarrow{\delta_!} !!A$ and $!G \xrightarrow{f} A$, then we can apply the functor '!' to f, to get $!!G \xrightarrow{!f} !A$ and if we precompose it with $!G \xrightarrow{\delta} !!G$ we get $!G \longrightarrow !A$, which shows the rule (!) is satisfied. But for this we are assuming that $!\Gamma \vdash A$ corresponds to a morphism $!G \rightarrow A$, and to know that we use the previous proposition as

$$!\Gamma = !G_1 \otimes !G_2 \otimes \ldots \otimes !G_k \cong !(G_1 \& G_2 \& \ldots G_k)$$

Next to show the soundness of (*contraction*) and (*weakening*) we show that $!A$ is a comonoid for the tensor product $\otimes_M$ in $M_N\mathbf{C}$. It is easy to calculate comonoids with respect to $\otimes_M$. They are objects, say A, equipped with a co-unit map to $I_M = (1 \overset{0}{\leftrightarrow} 1)$, $A \xrightarrow{\epsilon} I_M$ and a 'diagonal' map $A \xrightarrow{\delta} A \otimes A$, satisfying some commutative diagrams.

Then it is clear that $!A$ in $M_N\mathbf{C}$ is a comonoid with respect to $\otimes_M$, as we have morphisms $!A \xrightarrow{\epsilon} I$ and $!A \xrightarrow{\delta} !A \otimes !A$. Just check the diagrams:

$$\begin{tikzcd} U \arrow[d, "!"'] & X^{*U} \arrow[l, "!\alpha"'] \\ 1 & 1 \arrow[l, "0"] \arrow[u, "c_1"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[d, "\Delta"'] & & X^{*U} \arrow[ll, "!\alpha"'] \\ U \times U & & X^{*U\times U} \times X^{*U\times U} \arrow[ll, "!\alpha\otimes!\alpha"] \arrow[u, "c_2"'] \end{tikzcd}$$

Thus if $G \otimes !A \otimes !A \xrightarrow{f} B$ we can compose it with $!A \xrightarrow{\delta} !A \otimes !A$ to get $G \otimes !A \longrightarrow B$. And if $G \xrightarrow{g} B$ we can compose it with $!A \longrightarrow I$ to have $G \otimes !A \longrightarrow B$. □

4.5 Comparing Modalities

Similarly to what happen with the monoidal closed structures of $A_N\mathbf{C}$ and $M_N\mathbf{C}$, the exponential connectives $!_A$ and $!_M$ are comparable. In fact one could call $\mathcal{L}_3(A)$ the subset of the morphisms $\{\phi \mid \phi : U \rightarrow X^*\}$ such that $\forall u \in U$, $\forall\ \phi(u) \in X^*$, if $\phi(u) = \langle \phi(u)_1, \ldots, \phi(u)_n \rangle$ then $\alpha(u, \phi(u)_1) = \alpha(u, \phi(u)_2) = \ldots = \alpha(u, \phi(u)_n)$. We have then the following morphisms in $M_N\mathbf{C}$.

$$\begin{tikzcd} U \arrow[d, "1"'] & X^{*U} \arrow[l, "(!\alpha)_M"'] \\ U & \mathcal{L}_3(A) \arrow[l, "(!\alpha)_A"] \arrow[u, "i"'] \end{tikzcd}$$

5 Further Work

The linear negation $(\ )^\perp$ and the connective "*par*" of Linear Logic pose problems though in $M_N\mathbf{C}$, even if $\mathbf{C}$ is cartesian closed, even in **Sets**.

In our previous work, the bifunctor *par* was defined in the dialectica categories using disjunction of relations, but for multirelations it is not clear how to define 'disjunction' of natural numbers $m \vee n$. We know from the work in **GC** that $A \Box B$ should look like $(U^Y \times V^X \overset{\alpha\Box\beta}{\leftrightarrow} X \times Y)$, but the problem is to define a well-behaved morphism $\alpha\Box\beta$. One could try the following:

Conjecture 1 *Given objects $(U \overset{\alpha}{\leftrightarrow} X)$ and $(V \overset{\beta}{\leftrightarrow} Y)$ define their "par" as $(U^Y \times V^X \overset{\alpha\Box\beta}{\leftrightarrow} X \times Y)$, where the multirelation $\alpha\Box\beta$ is given by $\alpha\Box\beta(f, g, x, y) = max(\alpha(fy, x), \beta(gx, y))$*

The reason for the maximum of two natural numbers is that *max* looks a bit like logical *or*, if you think of of the truth table for $\vee$ and 0 means false. But then the identity for this *par* is $(1 \overset{0}{\leftrightarrow} 1)$, so (the linear logic constants) I and $\perp$ would coincide. Other possibilities for the map $\alpha\Box\beta$ are to take multiplication of natural numbers or their minimum.

But the reason none of these *pars* work is that they do not satisfy the weak distributive law [HdP'91] below:

$$a\Box(b \circ c) \leq (a\Box b) \circ c$$

In the case of multiplication the weak distributive law would give us $a \times (b+c) \cong a \times b + a \times c \leq a \times b + c$, only true if $a \leq 1$ a contradiction.

Also in the dialectica categories linear negation is defined in terms of linear implication into a dualizing object "$\perp$" which is the identity for *par*. If one considers, as in Lawvere's paper "∞" as an element of $\mathbf{N}$ that might induce a good choice for "$\perp$", but then linear negation is only defined for a very small class of objects.

In contrast to the situation described above, if one deals with the category $A_N\mathbf{C}$, one can define both 'par' and linear negation. But then you must have models of Classical Linear Logic, as the duality $A^{\perp\perp} \cong A$ is built into the category.

Conclusions

Much work remains to be done. On the mathematical side we want to try to obtain the right level of generality and to prove the existence of the modality '!' when the category $\mathbf{C}$ is only symmetric monoidal closed. Some work, with Martin Hyland is in progress, generalizing the construction of $M_N\mathbf{C}$ so that we model all of (full Intuitionistic) Linear Logic, see [HdP].

There are several questions as to how much of the work above can be done with ***non-symmetric*** monoidal closed categories. That would lead into non-commutative linear logic and there is some work in progress with Dominic Verity on it, as well as some other work on models of systems useful for linguistics purposes, see [dP'91].

On the applications side, there is some more work, besides [BGdP], with Carolyn Brown and Douglas Gurr on the connections between the models of Linear Logic that appear in Concurrency Theory. Finally, I would like to incorporate quantification into this general picture.

References

[Bar] M. BARR **$*$-Autonomous Categories**, LNM 752, Springer-Verlag, 1979.

[Bar] M. BARR *$*$-Autonomous Categories and Models of Linear Logic*, to appear in JAAP, 1991.

[B&G] C. BROWN and D. GURR *A Categorical Linear Framework for Petri Nets*, LICS'90.

[BGdP] C. BROWN, D. GURR and V.de PAIVA *General Petri Nets and Simulation Morphisms*, manuscript, Feb'91.

[Fla] R. C. FLAGG *A Generalised Logic*, manuscript, Fall'90.

[HdP] M. HYLAND and V.de PAIVA *Lineales*, manuscript Sept'90.

[PTJ] P. Johnstone **Topos Theory**, Academic Press.

[LSc] J. Lambek and P. Scott **Introduction to Higher-Order Categorical Logic**, CUP, 1986.

[Law] F.W. LAWVERE *Metric Spaces, Generalized Logic, and Closed Categories*, Rendiconti del Seminario Matematico e Fisico di Milano, 43 (1973).

[Laf] Y. LAFONT *From Linear Algebra to Linear Logic*, manuscript, November'88.

[LSt] Y. LAFONT and T. STREICHER *Games Semantics for Linear Logic*, to appear in LICS'91, Dec'90.

[deP] V.C.V. de PAIVA *A Dialectica-like Model of Linear Logic*, LNCS 389, 1989.

[deP] V.C.V. de PAIVA *A Dialectica Model of the Lambek Calculus*, manuscript, April'91.

[See] R. Seely *Linear Logic, $*$-Autonomous Categories and Cofree Coalgebras*, in Categories in Computer Science and Logic, AMS vol 92, eds. J. Gray and A. Scedrov, 1989.

[Ver] D. VERITY *Enrichments*, manuscript, Feb'91.

[Wins] G. WINSKEL *A Category of Labelled Petri Nets and Compositional Proof System*, LICS'88.

[Str] T. STREICHER *Adding modalities to* $\mathbf{GAME}_K$, personal communication at PSSL, March'90.

Appendix

Intuitionistic Linear Logic

We recall the axioms and rules of Intuitionistic Linear Logic, as in [Gir/L].
Axioms:

$$A \vdash A \quad (identity)$$
$$\vdash I$$
$$\Gamma \vdash 1 \qquad \Gamma, 0 \vdash A$$

Structural Rules:

$$\frac{\Gamma \vdash A}{\sigma\Gamma \vdash A}\ (permutation) \qquad \frac{\Gamma \vdash A \qquad A, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash B}\ (cut)$$

Logical Rules:

Multiplicatives:

$$(unit_l)\frac{\Gamma \vdash A}{\Gamma, I \vdash A}$$

$$(\otimes_l)\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} \qquad (\otimes_r)\frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}$$

$$(\multimap_l)\frac{\Gamma \vdash A \qquad \Gamma', B \vdash C}{\Gamma, \Gamma', A \multimap B \vdash C} \qquad (\multimap_r)\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

Additives:

$$(\&_r)\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B} \qquad (\&_l)\frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$$

$$(\oplus_l)\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} \qquad (\oplus_r)\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$$

Note that sequents have only one formula on the right-hand side of the turnstile.

Full Intuitionistic Linear Logic

We recall the axioms and rules of (Full Intuitionistic) Linear Logic.
Axioms:

$$A \vdash A \quad (identity)$$
$$\vdash I \qquad \bot \vdash$$
$$\Gamma \vdash 1, \Delta \qquad \Gamma, 0 \vdash \Delta$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \tau\Delta}\ (permutation) \qquad \frac{\Gamma \vdash A, \Delta \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ (cut)$$

Logical Rules:

Multiplicatives:

$$(unit_l)\frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta} \qquad (unit_r)\frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

$$(\otimes_l)\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad (\otimes_r)\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

$$(\Box_l)\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A\Box B \vdash \Delta, \Delta'} \qquad (\Box_r)\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A\Box B, \Delta}$$

$$(\multimap_l)\frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta', \Delta} \qquad (\multimap_r)\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (*)$$

Additives:

$$(\&_r)\frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A\&B, \Delta} \qquad (\&_l)\frac{\Gamma, A \vdash \Delta}{\Gamma, A\&B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A\&B \vdash \Delta}$$

$$(\oplus_l)\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad (\oplus_r)\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}$$

(*) Observe that in rule $(\multimap_r)$ we only deal with one formula on the right-hand side of the turnstile, according to our intuitionistic flavour of Linear Logic.