COMPUTER SCIENCE TRIPOS Part IA − 2021 − Paper 2

7 Discrete Mathematics (mpf23)

(a) Without using the Fundamental Theorem of Arithmetic, prove that

$$\gcd(c, ab) = 1 \iff (\gcd(c, a) = 1 \wedge \gcd(c, b) = 1)$$

for all positive integers a, b, c. [5 marks]

(b) Let $P(n)$ be a statement for n ranging over the set of positive integers $\mathbb{N}^+$.

(i) Prove that if
$$\forall m \in \mathbb{N}^+.\ P(m+1) \implies P(m)$$
then
$$\forall n \in \mathbb{N}^+.\ P(n+1) \implies (\forall k \in \mathbb{N}^+.\ k \leq n+1 \implies P(k))$$

[3 marks]

(ii) Prove that if
$$P(2) \wedge (\forall m \in \mathbb{N}^+.\ P(m) \Rightarrow P(2m)) \wedge (\forall m \in \mathbb{N}^+.\ P(m+1) \Rightarrow P(m))$$
then
$$\forall n \in \mathbb{N}^+.\ P(n)$$

[3 marks]

(c) Let $I = \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}$.

In each case below define a function from I to I that satisfies the stated properties. Your answer should justify that the criteria are met.

(i) Injective but not bijective. [3 marks]

(ii) Surjective but not bijective. [3 marks]

(iii) Bijective but not the identity. [3 marks]