9 Discrete Mathematics (MPF)

(*a*) Let r and s be solutions to the quadratic equation $x^2 - b\,x + c = 0$.

For $n \in \mathbb{N}$, define

$$d_0 = 0$$
$$d_1 = r - s$$
$$d_n = b\,d_{n-1} - c\,d_{n-2} \quad (n \geq 2)$$

Prove that $d_n = r^n - s^n$ for all $n \in \mathbb{N}$. [4 marks]

(*b*) Recall that a commutative monoid is a structure $(M, \mathtt{1}, *)$ where M is a set, $\mathtt{1}$ is an element of M, and $*$ is a binary operation on M such that

$$x * \mathtt{1} = x \quad , \quad x * y = y * x \quad , \quad (x * y) * z = x * (y * z)$$

for all x, y, z in M.

For a commutative monoid $(M, \mathtt{1}, *)$, consider the structure $(\mathcal{P}(M), I, \circledast)$ where $\mathcal{P}(M)$ is the powerset of M, I in $\mathcal{P}(M)$ is the singleton set $\{\mathtt{1}\}$, and $\circledast$ is the binary operation on $\mathcal{P}(M)$ given by

$$X \circledast Y = \{\, m \in M \mid \exists x \in X.\ \exists y \in Y.\ m = x * y \,\}$$

for all X and Y in $\mathcal{P}(M)$.

Prove that $(\mathcal{P}(M), I, \circledast)$ is a commutative monoid. [10 marks]

(*c*) Define a section-retraction pair to be a pair of functions $(s : A \to B, r : B \to A)$ such that $r \circ s = \mathrm{id}_A$.

(*i*) Prove that for every section-retraction pair (s, r), the section s is injective and the retraction r is surjective. [4 marks]

(*ii*) Exhibit two sets A and B together with an injective function $f : A \to B$ such that there is no function $g : B \to A$ for which (f, g) is a section-retraction pair. [2 marks]