7 Discrete Mathematics (MPF)

(a) (i) Calculate \( \gcd(144, 77) \), the greatest common divisor of 144 and 77, as an integer linear combination of 144 and 77. [4 marks]

(ii) What is the multiplicative inverse of 77 in \( \mathbb{Z}_{144} \) and the multiplicative inverse of 67 in \( \mathbb{Z}_{77} \)? [2 marks]

(iii) Describe all integers \( x \) that solve the following two congruences

\[
\begin{align*}
77 \cdot x &\equiv 1 \pmod{144} \\
67 \cdot x &\equiv 3 \pmod{77}
\end{align*}
\]

Indicate how one may calculate the least natural number solution to the above. [4 marks]

Justify your answers.

(b) For a string \( w \in \{1, 2\}^* \), let \( \sum(w) \in \mathbb{N} \) denote the sum of all the numbers in it. For instance, \( \sum(\varepsilon) = 0 \) for \( \varepsilon \) the null string, and \( \sum(1212) = 6 \).

For every \( n \in \mathbb{N} \), define \( S_n = \{ w \in \{1, 2\}^* \mid \sum(w) = n \} \). In particular, \( \varepsilon \in S_0 \) and \( 1212 \in S_6 \).

(i) List the elements of \( S_n \) for each \( n \in \{0, 1, 2, 3, 4, 5\} \). [2 marks]

(ii) What is the cardinality of \( S_n \) for each \( n \in \mathbb{N} \)? Prove your claim. [5 marks]

(iii) For all \( m, n \in \mathbb{N} \), define a bijective function

\[
((S_{m+1} \times S_{n+1}) \cup (S_m \times S_n)) \rightarrow S_{m+n+2}
\]

[3 marks]