5 Denotational Semantics (MPF)

For all PCF types \( \tau \) and all \( M \in \text{PCF}_{\tau \rightarrow \text{bool}} \), let \( M^\# \subseteq \llbracket \tau \rrbracket \) be defined as

\[
M^\# = \{ d \in \llbracket \tau \rrbracket \mid \|M\|_0(d) = \text{true} \}
\]

Indicate whether the following statements are true or false, respectively providing a proof or a counterexample. You may use any standard results provided that you state them clearly.

(a) For all PCF types \( \tau \) and all \( M, N \in \text{PCF}_{\tau \rightarrow \text{bool}} \), if \( M^\# \subseteq N^\# \) then \( \vdash M \leq_{\text{ctx}} N : \tau \). [5 marks]

(b) For all PCF types \( \tau \) and all \( M, N \in \text{PCF}_{\tau \rightarrow \text{bool}} \), if \( \vdash M \leq_{\text{ctx}} N : \tau \) then \( M^\# \subseteq N^\# \). [5 marks]

(c) For all PCF types \( \tau \) and all \( M, N \in \text{PCF}_{\tau \rightarrow \text{bool}} \), there exists \( P \in \text{PCF}_{(\tau \rightarrow \text{bool}) \rightarrow ((\tau \rightarrow \text{bool}) \rightarrow (\tau \rightarrow \text{bool}))} \) such that \( (P \ M \ N)^\# = M^\# \cap N^\# \). [5 marks]

(d) For all PCF types \( \tau \) and all \( M, N \in \text{PCF}_{\tau \rightarrow \text{bool}} \), there exists \( P \in \text{PCF}_{(\tau \rightarrow \text{bool}) \rightarrow ((\tau \rightarrow \text{bool}) \rightarrow (\tau \rightarrow \text{bool}))} \) such that \( (P \ M \ N)^\# = M^\# \cup N^\# \).

*Hint*: Consider the fact, for which you need not provide a proof, that there is no PCF-definable function \( f \in (\mathbb{B}_\bot \rightarrow (\mathbb{B}_\bot \rightarrow \mathbb{B}_\bot)) \) such that \( f \ \text{true} \bot = f \bot \text{true} = \text{true} \) and \( f \ \text{false} \bot \neq \text{true} \). [5 marks]