Advanced Algorithms (TMS)

(a) Explain the difference between PTAS and FPTAS, and give one example of a problem for which a FPTAS is known, and one example of a problem for which a PTAS is known but no FPTAS. [4 marks]

(b) We consider an extension of the MAX-3-CNF problem, called MAX-4-CNF problem, where we are given a 4-CNF formula with \( m \) clauses, e.g., \((x_1 \lor \overline{x_3} \lor x_4 \lor x_5) \land (\overline{x_1} \lor x_2 \lor x_3 \lor \overline{x_5}) \land \cdots\), and the goal is to find an assignment of the variables \( x_1, x_2, \ldots, x_n \) that satisfies as many clauses as possible.

(i) Design a randomised approximation algorithm and analyse its approximation ratio. (For full marks, the approximation ratio must be smaller than \(10/9\).) [4 marks]

(ii) Express the MAX-4-CNF problem as an integer program. [4 marks]

(iii) Based on the construction from Part (b)(ii) or otherwise, describe an algorithm that performs randomised rounding on the solution of a linear relaxation. [3 marks]

(iv) Analyse the expected approximation ratio of the algorithm from Part (b)(iii).

**Hint:** You may want to use the following two inequalities. Firstly, for any non-negative numbers \( a_1, a_2, \ldots, a_k \), we have

\[
\left( \prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{\sum_{i=1}^{k} a_i}{k}.
\]

Secondly, for any integer \( k \geq 2 \) and \( 0 \leq a \leq 1 \),

\[
1 - \left( 1 - \frac{a}{k} \right)^k \geq \left( 1 - \left( 1 - \frac{1}{k} \right)^k \right) \cdot a.
\]

[5 marks]