(a) (i) Suppose that \( F_X(x) \) is a distribution function. Show the inverse transform result, namely that, if \( U \) is a random variable uniformly distributed in the interval \((0,1)\) then

\[
X = F_X^{-1}(U)
\]

is a random variable with distribution function \( P(X \leq x) = F_X(x) \).

[4 marks]

(ii) Discuss the notion of a pseudo-random number generator for uniform random variables. Describe suitable algorithms for generating pseudo-random numbers.

[6 marks]

(iii) Using the inverse transform result in part (a)(i) derive a method to generate a stream of independent pseudo-random numbers from an exponential distribution with parameter \( \lambda > 0 \). What are the true mean and variance of these numbers in terms of \( \lambda \)?

[4 marks]

(b) (i) Suppose that you conduct a simulation experiment to estimate the mean, \( \mu \), of a random quantity \( X \) from a sample of \( n \) values \( X_1, X_2, \ldots, X_n \). How would you estimate \( \mu \)?

[2 marks]

(ii) Now suppose that your simulation also yields a sample of \( n \) values \( Y_1, Y_2, \ldots, Y_n \) of the random quantity \( Y \) where \( \mathbb{E}(Y) = \mu_Y \) is a known number. How would you use the method of control variates to improve your estimator of \( \mu \)? Your answer should mention all quantities that may need to be estimated and in what way you will improve the estimation of \( \mu \).

[4 marks]