10 Quantum Computing (AD)

Let $f : \{0, 1\}^2 \to \{0, 1\}$ be a Boolean function of two inputs. Let $U_f$ be the implementation of $f$ as a unitary operator on 3 qubits defined by:

$$U_f|x\rangle|y\rangle|z\rangle = |x\rangle|y\rangle|z \oplus f(x, y)\rangle,$$

where $\oplus$ denotes the exclusive-or operation, and $|x\rangle|y\rangle|z\rangle$ is any computational basis state.

Consider the following circuit (a two-qubit version of the Deutsch-Josza circuit) in which $X$ denotes a NOT gate, $H$ denotes a Hadamard gate and $M$ is a two-qubit measurement in the computational basis.

(a) Show that if $f$ is a constant function, the outcome of the measurement $M$ is 00 with probability 1. [6 marks]

(b) Show that if $f$ is the XOR function, the outcome of the measurement $M$ is 11 with probability 1. [6 marks]

(c) What are the probabilities of $M$ measuring 00 and 11 respectively, if $f$ is the Boolean AND function? [8 marks]