8 Hoare Logic and Model Checking (AM)

This question considers a language $L$ which has integer variables $V$, arithmetic expressions $E$ and boolean expressions $B$, along with commands $C$ of the forms $V := E$ (assignment), $C; C'$ (sequencing), $\text{IF } B \text{ THEN } C \text{ ELSE } C'$ (conditional) and $\text{WHILE } B \text{ DO } C$ (iteration).

(a) Explain the syntax of the Hoare-logic partial-correctness formula $\{P\} C \{Q\}$ and give a careful definition in English of when it is valid, that is, when $\models \{P\} C \{Q\}$. [2 marks]

(b) How does the definition of validity for the total-correctness formula $[P] C [Q]$ differ? [1 mark]

(c) Preconditions and postconditions in $\{P\} C \{Q\}$ often make use of logical or auxiliary variables $v$ in addition to program variables $V$. Explain why this is useful illustrating your answer with a command $C$ which satisfies $\{T\} C \{R = X + Y\}$ but not $\{X = x \land Y = y\} C \{R = x + y\}$. [3 marks]

(d) Give the axioms and rules of an inference system $\vdash \{P\} C \{Q\}$ for Hoare logic. [4 marks]

(e) Are your rules sound? To what extent are they complete? [2 marks]

(f) Give a formal proof, using your inference system, of
$\{X = x \land Y = 3\} \text{X:=X+1} \{X - 1 = x \land Y < 10\}$. [2 marks]

(g) Consider the command $C$ given by $\text{WHILE } X > 0 \text{ DO } (X := X - 1; Y := Y + 3)$, and let $P$ be the precondition $X = x \land Y = y \land x \geq 0$. Give the strongest postcondition $Q$ that you can establish. Give any invariant necessary to prove $\{P\} C \{Q\}$ for your $Q$. Explain briefly how the structure of the proof relates to the structure of $C$. [6 marks]