7 Denotational Semantics (MPF)

(a) (i) Define the notion of continuous function between domains. [2 marks]

(ii) Let \( \mathcal{P}(\mathbb{N}^2) \) be the domain of all subsets of pairs of natural numbers ordered by inclusion. Show that the function \( f : \mathcal{P}(\mathbb{N}^2) \to \mathcal{P}(\mathbb{N}^2) \) given by

\[
 f(S) = \{ (1, 1) \} \cup \{ (x + 1, x \cdot y) \in \mathbb{N}^2 \mid (x, y) \in S \} \quad (S \subseteq \mathbb{N}^2)
\]

is continuous. [3 marks]

(b) (i) State Tarski’s fixed point theorem for a continuous endofunction on a domain. [2 marks]

(ii) Give a concrete explicit description of the fixed point \( \text{fix}(f) \subseteq \mathbb{N}^2 \) of the continuous function \( f \) in Part (a)(ii). Briefly justify your answer. [3 marks]

(c) (i) Define the notion of an admissible subset of a domain. [2 marks]

(ii) Let \( P \subseteq \mathcal{P}(\mathbb{N}^2) \) be defined as \( P = \{ S \subseteq \mathbb{N}^2 \mid \forall (x, y) \in S. \log y \leq x \cdot \log x \} \). Show that \( P \) is an admissible subset of the domain \( \mathcal{P}(\mathbb{N}^2) \). [3 marks]

(d) (i) State Scott’s fixed point induction principle. [2 marks]

(ii) Use Scott’s fixed point induction principle to show that \( \text{fix}(f) \in P \) for \( f \) the continuous function in Part (a)(ii) and \( P \) the admissible subset of the domain \( \mathcal{P}(\mathbb{N}^2) \) in Part (c)(ii). [3 marks]