8 Mathematical Methods for Computer Science (RJG)

(*a*) (*i*) Consider a random variable X with moment generating function $M_X(t)$. State Chernoff's bound for the probability $\mathbb{P}(X \geq a)$ where a is a constant. [2 marks]

(*ii*) If $X \sim \text{Binomial}(n, p)$ apply Chernoff's bound to X and minimize the upper bound over the values $t > 0$ to show that for $np < a < n$

$$\mathbb{P}(X \geq a) \leq \left(\frac{np}{a}\right)^a \left(\frac{n(1-p)}{n-a}\right)^{n-a} .$$

[8 marks]

(*b*) An online service company receives n tasks per unit time and wishes to serve these tasks using m servers. The allocation of the tasks to the servers is by a randomized load balancing strategy that assigns each of the n tasks independently and uniformly to one of the m servers. Each server can serve up to and including t tasks per unit time without becoming overloaded. Let X_i for $i = 1, 2, \ldots, m$ be the random number of tasks assigned to the i^{th} server in a given unit of time.

(*i*) What is the marginal distribution of X_i for each $i = 1, 2, \ldots, m$? [2 marks]

(*ii*) State whether or not the random variables X_i for $i = 1, 2, \ldots, m$ are mutually independent. Justify your result. [3 marks]

(*iii*) Let $Y_m = \max\{X_1, X_2, \ldots, X_m\}$ and show that

$$\mathbb{P}(Y_m \geq a) \leq m\mathbb{P}(X_i \geq a) \qquad i = 1, 2, \ldots, m .$$

You may assume without proof that if $A_1, A_2, \ldots, A_r$ are random events then $\mathbb{P}(\cup_{i=1}^r A_i) \leq \sum_{i=1}^r \mathbb{P}(A_i)$. [2 marks]

(*iv*) The company asks your advice about a suitable number of servers to rent so that the probability that at least one of the servers is overloaded in a given unit of time is no greater than 0.01. Determine an expression for the least value of m such that the stated criterion is met. [3 marks]