7 Mathematical Methods for Computer Science (JGD)

(a) (i) Express $W$, the primitive $N^{th}$ root of unity, as a complex exponential. [2 marks]

(ii) Express the $N$-point real-valued discrete sequence $f[n] = \cos(2\pi n/N)$ for $n = 1, 2, 3, \ldots, N$ in terms of $W$. [3 marks]

(iii) Using a vector sum diagram in the complex plane, show how elements of the real-valued discrete sequence $f[n]$ are represented as a sum of complex numbers related to $W$, each having unit length. Construct your diagram for the particular case of integer $n = N/8$. [2 marks]

(b) A zero-centred pulse function $F(\omega)$ in the frequency domain $\omega$, having unit area $F(\omega) = 1/2$ for $\omega \in [-1, +1]$, and $F(\omega) = 0$ for $|\omega| > 1$, represents one ideal low-pass filter.

\[1/2\]
\[F(\omega)\]
\[-1\]
\[1\]
\[\omega\]

(i) Derive its inverse Fourier transform $f(x)$. [4 marks]

(ii) Sketch a plot of this function and specify the roots of $f(x) = 0$. [2 marks]

(c) Let $f(x)$ be any real-valued function whose Fourier transform $F(\omega)$ exists. Show that $F(\omega)$ has the property of Hermitian symmetry $F(-\omega) = F(\omega)$, and comment on the computational benefits that result from this property.

\textit{Hint:} Represent $f(x)$ as the sum of an even function $f_e(x)$ plus an odd function $f_o(x)$, where

\[f_e(x) = \frac{1}{2}(f(x) + f(-x))\]
\[f_o(x) = \frac{1}{2}(f(x) - f(-x))\]

and then consider the Fourier transform of $f(x) = f_e(x) + f_o(x)$. You may invoke known properties of even- and odd-symmetric functions without proof. [7 marks]