## 9 Discrete Mathematics (MPF)

(*a*) Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers $a, b, c$,

$$\gcd(a, c) = 1 \implies \big(\gcd(a \cdot b, c) \mid b \wedge \gcd(a \cdot b, c) = \gcd(b, c)\big)$$

You may use any other standard results provided that you state them clearly. [6 marks]

(*b*) Prove that for all disjoint sets $X$ and $Y$ (that is, such that $X \cap Y = \emptyset$),

$$\mathcal{P}(X \cup Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$$

You may use any standard results provided that you state them clearly. [6 marks]

(*c*) (*i*) For an alphabet $\Sigma$ and a language $L \subseteq \Sigma^*$, define the language $F(L) \subseteq \Sigma^*$ as

$$F(L) \stackrel{\text{def}}{=} \{\, awa \in \Sigma^* \mid a \in \Sigma \wedge w \in L \,\}$$

Prove that for all $L_1, L_2 \subseteq \Sigma^*$, $F(L_1 \cup L_2) = F(L_1) \cup F(L_2)$. [2 marks]

(*ii*) Let $\mathrm{Pal} \subseteq \Sigma^*$ be the language of palindromes (i.e. strings that read the same backwards as forwards) defined as $\mathrm{Pal} \stackrel{\text{def}}{=} \{\, w \in \Sigma^* \mid \mathrm{rev}(w) = w \,\}$ where rev is the unique function $\Sigma^* \to \Sigma^*$ such that $\mathrm{rev}(\varepsilon) = \varepsilon$, $\mathrm{rev}(a) = a$ for all $a \in \Sigma$, and $\mathrm{rev}(w_1 w_2) = \mathrm{rev}(w_2)\,\mathrm{rev}(w_1)$ for all $w_1, w_2 \in \Sigma^*$.

Prove that for all $L \subseteq \Sigma^*$, $L \subseteq \mathrm{Pal} \implies F(L) \subseteq \mathrm{Pal}$. [2 marks]

(*iii*) For $k \in \mathbb{N}$, let $\mathrm{Pal}_k \subseteq \mathrm{Pal}$ be the language of palindromes of length $k$; that is, $\mathrm{Pal}_k \stackrel{\text{def}}{=} \{w \in \mathrm{Pal} \,\big|\, |w| = k\}$.

Recalling that $F^0(X) \stackrel{\text{def}}{=} X$ and, for $k \in \mathbb{N}$, $F^{k+1}(X) \stackrel{\text{def}}{=} F(F^k(X))$, prove that:

(A) for all $n \in \mathbb{N}$, $F^n(\mathrm{Pal}_0) = \mathrm{Pal}_{2n}$, [2 marks]

(B) for all $n \in \mathbb{N}$, $F^n(\mathrm{Pal}_1) = \mathrm{Pal}_{2n+1}$. [2 marks]

It follows that $\bigcup_{n \in \mathbb{N}} F^n(\mathrm{Pal}_0 \cup \mathrm{Pal}_1) = \mathrm{Pal}$, though you are not required to prove this.