7 Discrete Mathematics (MPF)

(a) Let $N_{\geq 2} \overset{\text{def}}{=} \{ k \in \mathbb{N} \mid k \geq 2 \}$.

Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers $m$ and $n$,

$$\text{gcd}(m, n) = 1 \iff \neg(\exists k \in N_{\geq 2}. k \mid m \land k \mid n)$$

You may use any other standard results provided that you state them clearly. [6 marks]

(b) Recall that, for $i, j \in \mathbb{N}$,

$$\binom{i}{j} \overset{\text{def}}{=} \left\{ \begin{array}{ll} 0, & \text{if } i < j \\
\frac{i!}{j!(i-j)!}, & \text{if } i \geq j \end{array} \right.$$  

(i) Show that for all $m < l$ in $\mathbb{N}$,

$$\binom{l}{m+1} + \binom{l}{m} = \binom{l+1}{m+1}$$  

[2 marks]

(ii) Prove that

$$\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. 0 \leq m \leq n \implies \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$  

[6 marks]

(c) Let $U$ be a set and let $F : \mathbb{N} \times \mathbb{N} \rightarrow \mathcal{P}(U)$ be a function such that for all $i, i', j, j' \in \mathbb{N}$, if $i \leq i'$ and $j \leq j'$ then $F(i, j) \subseteq F(i', j')$ in $\mathcal{P}(U)$.

Prove that

$$\bigcup_{i \in \mathbb{N}} \left( \bigcup_{j \in \mathbb{N}} F(i, j) \right) = \bigcup_{k \in \mathbb{N}} F(k, k)$$

(Recall that $x \in \bigcup_{i \in L} X_i \iff \exists l \in L. x \in X_l$)  

[6 marks]