4 Computer Systems Modelling (RJG)

Consider a birth death process $X(t)$ for $t \geq 0$ with states $0, 1, 2, \ldots$ and where the birth rate is λ_j in states $j = 0, 1, \ldots$ and the death rate is μ_j in states $j = 1, 2, \ldots$

(*a*) Draw the state space diagram for the birth death process. [2 marks]

(*b*) Derive the Chapman-Kolmogorov differential equations for the birth death process in terms of $P_j(t) = \mathbb{P}(X(t) = j)$ for $j = 0, 1, \ldots$ and $t \geq 0$. [4 marks]

(*c*) State the detailed balance conditions for an equilibrium distribution $p_j = P_j(t)$ for $j = 0, 1, \ldots$ to solve the Chapman-Kolmogorov equations with $\frac{dP_j(t)}{dt} = 0$ and determine a further condition to ensure the existence of the equilibrium distribution. Derive an expression for equilibrium distribution p_j when your further condition holds. [4 marks]

(*d*) Describe a birth death process model for an $M/M/K$ system in the limit as the number of servers $K \to \infty$. Draw the state space diagram, give the birth and death rates and derive the equilibrium distribution stating whether there are any conditions for its existence. [5 marks]

(*e*) Now consider the Chapman-Kolmogorov equations derived in part (*b*) in the special case of a pure birth process with constant birth rates $\lambda_j = \lambda$ for $j = 0, 1, \ldots$ and zero death rates $\mu_j = 0$ for $j = 1, 2, \ldots$ Suppose that the process starts in state 0 at time $t = 0$ (that is $P_0(0) = 1$ and $P_j(0) = 0$ for $j = 1, 2, \ldots$). Thus $X(t)$ is the number of events in a Poisson process of rate λ. Determine the Chapman-Kolmogorov differential equations for the time-dependent solution $P_j(t)$ for $j = 0, 1, 2, \ldots$ and $t \geq 0$ in this case and solve for the explicit solutions $P_j(t)$ obeying the initial conditions. [5 marks]