9 Discrete Mathematics (AMP)

(*a*) The subset S of $\mathbb{N} = \{0, 1, 2, \ldots\}$ is inductively defined by the following axiom and rules, where n ranges over $\mathbb{N}$:

$$\frac{}{1} \qquad \frac{n}{2n} \qquad \frac{n}{3n} \qquad \frac{n+5}{n}$$

(*i*) State the principle of Rule Induction associated with this set of axioms and rules. [4 marks]

(*ii*) Use Rule Induction to prove that no element of S is divisible by 5. [4 marks]

(*iii*) Is 0 an element of S? Justify your answer. [1 mark]

(*b*) State the principle of Mathematical Induction. [2 marks]

(*c*) For sets X and Y of strings over an alphabet Σ, let XY denote the set $\{uv \mid u \in X \text{ and } v \in Y\}$ of all concatenations of a string in X followed by a string in Y. For $n \in \mathbb{N}$, let X^n be given by: $X^0 = \{\varepsilon\}$ (where ε denotes the null string) and $X^{n+1} = XX^n$. Let $X^* = \bigcup_{n \geq 0} X^n$.

Suppose $X, Y, Z \subseteq \Sigma^*$ satisfy $Z = XZ \cup Y$.

(*i*) Prove by Mathematical Induction that $\forall n \in \mathbb{N}.\, X^n Y \subseteq Z$ and deduce that $X^* Y \subseteq Z$. [4 marks]

(*ii*) Suppose further that $\varepsilon \notin X$. By considering the property of $n \in \mathbb{N}$ given by $\forall w \in Z.\, |w| \leq n \Rightarrow w \in X^* Y$, or otherwise, use Mathematical Induction to prove that $Z \subseteq X^* Y$. [5 marks]