8 Discrete Mathematics (MPF)

(*a*) Let $\#X$ denote the cardinality of a set X.

Define a unary predicate P for which the statement

$$\forall \text{ sets } X. \left[P(\#X) \iff (\forall \text{ sets } A, B.\ A \times X = B \times X \implies A = B\,) \right]$$

holds. [1 mark]

Prove the statement for your given predicate P. [4 marks]

(*b*) For sets X, Y, Z, let $(X \Rightarrow Y)$ denote the set of functions from X to Y and let $\mathcal{P}(Z)$ be the powerset of Z.

For sets A and B, consider the function

$$(\cdot)^\sharp : \big(A \Rightarrow \mathcal{P}(B)\big) \longrightarrow \big(\mathcal{P}(A) \Rightarrow \mathcal{P}(B)\big)$$

given, for all $f \in \big(A \Rightarrow \mathcal{P}(B)\big)$ and $X \in \mathcal{P}(A)$, by

$$f^\sharp(X) = \bigcup_{a \in X} f(a)$$

Show that for all $g \in \big(\mathcal{P}(A) \Rightarrow \mathcal{P}(B)\big)$ there exists $f \in \big(A \Rightarrow \mathcal{P}(B)\big)$ such that $f^\sharp = g$ iff, for all $\mathcal{F} \subseteq \mathcal{P}(A)$, $g\big(\bigcup_{X \in \mathcal{F}} X\big) = \bigcup_{X \in \mathcal{F}} g(X)$. [6 marks]

(*c*) For sets S and A, let $\mathrm{Bij}(S, S)$ be the set of bijections from S to S, let $\mathrm{Inj}(S, A)$ be the set of injections from S to A, and let $\mathcal{P}_S(A) = \big\{ X \subseteq A \mid X \cong S \big\}$ be the set of subsets of A that are in bijection with S.

(*i*) Prove that the relation $\approx\ \subseteq \mathrm{Inj}(S, A) \times \mathrm{Inj}(S, A)$ defined, for all $f, g \in \mathrm{Inj}(S, A)$, by

$$f \approx g \iff \exists\, h \in \mathrm{Bij}(S, S)\,.\ f = g \circ h$$

is an equivalence relation. [3 marks]

(*ii*) Define a bijection

$$\mathrm{Inj}(S, A)_{/\approx} \longrightarrow \mathcal{P}_S(A)$$

You need not prove your function is bijective, but you should explain why your mapping is well defined. [6 marks]