4 Computer Systems Modelling (RJG)

Consider a single server queue with customer arrivals occuring at times of a Poisson process of rate λ. Customers are served with independent exponential service times which have mean $1/\hat{\mu}$ if there are less than k customers present and mean $1/\mu$ if there are k or more customers present and where k is fixed. The number present, n, can be modeled as a birth-death process with the birth rates $\lambda_n = \lambda$ for each $n = 0, 1, 2, \ldots$ and state-dependent death rates

$$\mu_n = \begin{cases} \hat{\mu} & 1 \leq n < k \\ \mu & n \geq k \end{cases} .$$

Write $\hat{\rho} = \lambda/\hat{\mu}$ and $\rho = \lambda/\mu$ and assume that $\rho < 1$.

(*a*) Find an expression for π_n the probability of being in state n under the equilibrium distribution which you may assume to exist. [4 marks]

(*b*) Show that if $\hat{\mu} = \mu$ then your result for π_n in part (*a*) coincides with the case of a M/M/1 queue, namely $\rho^n(1-\rho)$. [1 mark]

(*c*) Find an expression for L the expected number of customers in the system. [6 marks]

(*d*) Show that

$$L_q = L - (1 - \pi_0)$$

where L_q is the expected number of customers in the queue waiting for service. [2 marks]

(*e*) Find expressions for the expected time, W, that a customer spends in the system and the expected time, W_q, that a customer spends waiting for service. [2 marks]

(*f*) Consider an example where customers arrive according to a Poisson process with a mean inter-arrival time of 30 minutes. Suppose that the service times are exponential with mean 40 minutes if there are no customers waiting but have mean 20 minutes if there are any customers waiting. Compute π_0, L, L_q, W and W_q. [5 marks]