7 Mathematical Methods for Computer Science (JGD)

(a) For vectors $u, v \in V$ in linear space $V = \mathbb{R}^n$ with $u = (u_1, u_2, \ldots, u_n)$, define the Euclidean norm $||u||$, and state the triangle inequality for $||u + v||$. [2 marks]

(b) Define cyclical convolution of two periodic sequences $f[n]$ and $g[n]$. [2 marks]

(c) If $\Psi(x)$ is a generating (or “mother”) wavelet, give the dyadic shifting and scaling operations that generate her “daughter” wavelets $\Psi_{jk}(x)$ in terms of dilates $j$ and translates $k$ of $\Psi(x)$. [2 marks]

(d) Why is the dyadic property of wavelets useful for analysing naturally-arising data that often exhibits self-similarity across scales? [2 marks]

(e) Derive the Fourier series of a periodic triangle wave, $f(x) = |x|$ for $x \in [-\pi, \pi]$ [4 marks]

(f) The Modulation Theorem asserts that if $f(x)$ has Fourier transform $F(\omega)$, then modulating $f(x)$ at frequency $c$ (multiplying it by $e^{icx}$) simply shifts its transform up by $c$ to become $F(\omega - c)$. Prove this, and explain one important practical application of this property. [4 marks]

(g) Show how Fourier methods enable solution of differential equations such as the following, in which the function $g(x)$ is known (hence its Fourier transform $G(\omega)$ can be computed), and $a, b, c$ are constant coefficients. Derive an expression for $f(x)$ that solves this differential equation.

$$a \frac{d^2 f(x)}{dx^2} + b \frac{df(x)}{dx} + cf(x) = g(x)$$