6 Discrete Mathematics II (MPF)

Let $R \subseteq U \times U$ be a relation on a set U.

(*a*) Let $R^\dagger \subseteq U \times U$ be the relation inductively defined by the rules

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(a,b) \quad (b,c)}{(a,c)}$$

and let $R^\bullet \subseteq U \times U$ be the relation inductively defined by the rules

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(b,c)}{(a,c)}\ (a,b) \in R$$

Either prove or disprove the following statements.

(*i*) $R^\bullet \subseteq R^\dagger$ [4 marks]

(*ii*) $R^\dagger \subseteq R^\bullet$ [4 marks]

(*b*) Let $R^\diamond \subseteq U \times U$ be the relation inductively defined by the rules

$$\frac{}{(a,b)}\ (a,b) \in R \qquad\qquad \frac{(b,c)}{(a,d)}\ (a,b),(c,d) \in R$$

Either prove or disprove the following statements.

(*i*) $R^\diamond \subseteq \bigcup_{n \in \mathbb{N}_0} R^{2n+1}$ [4 marks]

(*ii*) $\bigcup_{n \in \mathbb{N}_0} R^{2n+1} \subseteq R^\diamond$ [4 marks]

(*iii*) $\left(R^\diamond\right)^{-1} = \left(R^{-1}\right)^\diamond$ [4 marks]

You may assume without proof that for each $n \in \mathbb{N}_0$, the relation $R^n \subseteq U \times U$ satisfies $R \circ R^n = R^{n+1} = R^n \circ R$ and $(R^n)^{-1} = (R^{-1})^n$.