7 Mathematical Methods for Computer Science (JGD)

(*a*) Define linear independence and linear dependence for the set of vectors $\{v_1, v_2, \ldots, v_n\}$ of a vector space V over a field $\mathbb{F}$ of scalars $a_1, a_2, \ldots, a_n \in \mathbb{F}$.
[4 marks]

(*b*) Using the Euclidean norm on an inner product space $V = \mathbb{R}^3$, for the following vectors $u, v \in V$ whose span is a linear subspace of V,

$$u = \left(\frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}}, \ \frac{1}{\sqrt{3}} \right)$$

$$v = \left(\sqrt{3}, \ -\frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} \right)$$

demonstrate whether u, v form an *orthogonal system*, and also whether they form an *orthonormal system.* [4 marks]

(*c*) Using a diagram in the complex plane showing the N^{th} roots of unity, explain why all the values of complex exponentials that are needed for computing the Discrete Fourier transform of N data points are powers of a primitive N^{th} root of unity (circled here for $N = 16$), and explain why such factorisation greatly reduces the number of multiplications required in a Fast Fourier transform.
[4 marks]

(*d*) For the function $f(x) = e^{-a|x|}$ with $a > 0$, derive its Fourier transform $F(\omega)$.
[4 marks]

(*e*) For a function $f(x)$ whose Fourier transform is $F(\omega)$, what is the Fourier transform of $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ with respect to x? Explain how Fourier methods make it possible to define non-integer orders of derivatives, and name one scientific field in which it is useful to take half-order derivatives.
[4 marks]