7 Mathematical Methods for Computer Science (JGD)

(a) Define linear independence and linear dependence for the set of vectors \( \{v_1, v_2, \ldots, v_n\} \) of a vector space \( V \) over a field \( \mathbb{F} \) of scalars \( a_1, a_2, \ldots, a_n \in \mathbb{F} \).

(b) Using the Euclidean norm on an inner product space \( V = \mathbb{R}^3 \), for the following vectors \( u, v \in V \) whose span is a linear subspace of \( V \),

\[
\begin{align*}
  u &= \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \\
  v &= \left( \sqrt{3}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2} \right)
\end{align*}
\]

demonstrate whether \( u, v \) form an orthogonal system, and also whether they form an orthonormal system.

(c) Using a diagram in the complex plane showing the \( N \)th roots of unity, explain why all the values of complex exponentials that are needed for computing the Discrete Fourier transform of \( N \) data points are powers of a primitive \( N \)th root of unity (circled here for \( N = 16 \)), and explain why such factorisation greatly reduces the number of multiplications required in a Fast Fourier transform.

(d) For the function \( f(x) = e^{-a|x|} \) with \( a > 0 \), derive its Fourier transform \( F(\omega) \).

(e) For a function \( f(x) \) whose Fourier transform is \( F(\omega) \), what is the Fourier transform of \( f^{(n)}(x) \), the \( n \)th derivative of \( f(x) \) with respect to \( x \)? Explain how Fourier methods make it possible to define non-integer orders of derivatives, and name one scientific field in which it is useful to take half-order derivatives.