Topics in Concurrency

This question concerns CTL$^{-}$, a variant of CTL in which assertions are of the form:

$$A := T \mid A_0 \lor A_1 \mid A_0 \land A_1 \mid \neg A \mid \langle \lambda \rangle A \mid \langle - \rangle A \mid AG A \mid EG A$$

where $\lambda$ ranges over action names. Recall that a path $\pi$ from state $s$ is a maximal sequence of states $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ such that $s = \pi_0$ and $\pi_i \rightarrow \pi_{i+1}$ for all $i$. The logical connectives have the standard interpretation and $T$ represents “true”.

The interpretation of the modalities is:

- $s \models \langle \lambda \rangle A$ iff there exists $s'$ such that $s \xrightarrow{\lambda} s'$ and $s' \models A$
- $s \models \langle - \rangle A$ iff there exists $s'$ such that $s \xrightarrow{} s'$ and $s' \models A$
- $s \models EG A$ iff for some path $\pi$ from $s$, we have $\pi_i \models A$ for all $i$
- $s \models AG A$ iff for all paths $\pi$ from $s$, we have $\pi_i \models A$ for all $i$

(a) What is the interpretation of the CTL modality $E[\langle a \rangle A_0 U A_1]$? How can it be used to express the CTL$^{-}$ modality $AG A$? [4 marks]

(b) Consider the following three formulae:

$$A_1 : AG \langle a \rangle T \quad A_2 : \neg AG \neg \langle b \rangle T \quad A_3 : \neg EG \langle b \rangle T$$

(i) For each of the following two transition systems, state which of $A_1$, $A_2$ and $A_3$ are satisfied in the initial state.

- $P_1$:

  ![Transition System 1]

  [4 marks]

- $P_2$:

  ![Transition System 2]

  [4 marks]

(ii) Draw a transition system with an initial state that satisfies $A_1 \land A_2 \land A_3$. [3 marks]

(c) Give a modal-$\mu$ formula that corresponds to the CTL$^{-}$ formula $AG \langle a \rangle T$. [2 marks]

(d) Let the function $\varphi$ on sets of states of a transition system be defined as:

$$\varphi(X) \overset{\text{def}}{=} \langle a \rangle T \lor \langle - \rangle X$$

Show by induction on $n \geq 1$ that

- $s \models \varphi^n(\emptyset)$ iff there exists $m \leq n$ and states $s_1, \ldots, s_m, s'$ such that $s = s_1 \xrightarrow{} \cdots \xrightarrow{} s_m \xrightarrow{a} s'$

Deduce that $s \models \mu X. (A) T \lor \langle - \rangle X$ in a finite-state transition system if, and only if, $s \models \neg AG \neg \langle a \rangle T$. [7 marks]